Spectral properties of random non-self-adjoint operators

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PROPRIÉTÉS SPECTRALES DES OPÉRATEURS NON-AUTO-ADJOINTS ALÉATOIRES

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RÉSUMÉ / ABSTRACT

Résumé

Dans cette thèse, nous nous intéressons aux propriétés spectrales des opérateurs non-auto-adjoints aléatoires. Nous allons considérer principalement les cas des petites perturbations aléatoires de deux types des opérateurs non-auto-adjoints suivants:

1. une classe d'opérateurs non-auto-adjoints h-différentiels P_h, introduite par M. Hager [32], dans la limite semiclassique ($h \to 0$);

2. des grandes matrices de Jordan quand la dimension devient grande ($N \to \infty$).

Dans le premier cas nous considérons l'opérateur P_h soumis à de petites perturbations aléatoires. De plus, nous imposons que la constante de couplage δ vérifie $\mathrm{e}^{-1/Ch} \leq \delta \ll h^{\kappa}$, pour certaines constantes $C, \kappa > 0$ choisies assez grandes. Soit Σ l'adhérence de l'image du symbole principal de P_h. De précédents résultats par M. Hager [32], W. Bordeaux-Montrieux [4] et J. Sjöstrand [67] montrent que, pour le même opérateur, si l'on choisit $\delta \gg \mathrm{e}^{-1/Ch}$, alors la distribution des valeurs propres est donnée par une loi de Weyl jusqu'à une distance $\gg (-h \ln \delta h)^{\frac{2}{3}}$ du bord de Σ.

Nous étudions la mesure d'intensité à un et à deux points de la mesure de comptage aléatoire des valeurs propres de l'opérateur perturbé. En outre, nous démontrons des formules h-asymptotiques pour les densités par rapport à la mesure de Lebesgue de ces mesures qui décrivent le comportement d'un seul et de deux points du spectre dans Σ. En étudiant la densité de la mesure d'intensité à un point, nous prouvons qu'il y a une loi de Weyl à l'intérieur du pseudospectre, une zone d'accumulation des valeurs propres dûe à un effet tunnel près du bord du pseudospectre suivi par une zone où la densité décroît rapidement.

En étudiant la densité de la mesure d'intensité à deux points, nous prouvons que deux valeurs propres sont répulsives à distance courte et indépendantes à grande distance à l'intérieur de Σ.

Dans le deuxième cas, nous considérons des grands blocs de Jordan soumis à des petites perturbations aléatoires gaussiennes. Un résultat de E.B. Davies et M. Hager [16] montre que lorsque la dimension de la matrice devient grande, alors avec probabilité proche de 1, la plupart des valeurs propres sont proches d'un cercle. De plus, ils donnent une majoration logarithmique du nombre de valeurs propres à l'intérieur de ce cercle.

Nous étudions la répartition moyenne des valeurs propres à l'intérieur de ce cercle et nous en donnons une description asymptotique précise. En outre, nous démontrons que le terme principal de la densité est donné par la densité par rapport à la mesure de Lebesgue de la forme volume induite par la métrique de Poincaré sur la disque $D(0,1)$.

Mots-clefs Théorie spectrale; Opérateurs non-auto-adjoints; Opérateurs différentiels semiclassique; Perturbations aléatoires.

SPECTRAL PROPERTIES OF RANDOM NON-SELF-ADJOINT OPERATORS

Abstract

In this thesis we are interested in the spectral properties of random non-self-adjoint operators. We are going to consider primarily the case of small random perturbations of the following two types of operators:

1. a class of non-self-adjoint h-differential operators P_h, introduced by M. Hager [32], in the semiclassical limit ($h \to 0$);

2. large Jordan block matrices as the dimension of the matrix gets large ($N \to \infty$).

In case 1 we are going to consider the operator P_h subject to small Gaussian random perturbations. We let the perturbation coupling constant δ be $\mathrm{e}^{-1/Ch} \leq \delta \ll h^\kappa$, for constants $C, \kappa > 0$ suitably large. Let Σ be the closure of the range of the principal symbol. Previous results on the same model by M. Hager [32], W. Bordeaux-Montrieux [4] and J. Sjöstrand [67] show that if $\delta \gg \mathrm{e}^{-1/Ch}$ there is, with a probability close to 1, a Weyl law for the eigenvalues in the interior of the pseudospectrum up to a distance $\gg (-h \ln \delta h)^{\frac{2}{3}}$ to the boundary of Σ.

We will study the one- and two-point intensity measure of the random point process of eigenvalues of the randomly perturbed operator and prove h-asymptotic formulae for the respective Lebesgue densities describing the one- and two-point behavior of the eigenvalues in Σ. Using the density of the one-point intensity measure, we will give a complete description of the average eigenvalue density in Σ describing as well the behavior of the eigenvalues at the pseudospectral boundary. We will show that there are three distinct regions of different spectral behavior in Σ:

The interior of the of the pseudospectrum is solely governed by a Weyl law, close to its boundary there is a strong spectral accumulation given by a tunneling effect followed by a region where the density decays rapidly.

Using the h-asymptotic formula for density of the two-point intensity measure we will show that two eigenvalues of randomly perturbed operator in the interior of Σ exhibit close range repulsion and long range decoupling.

In case 2 we will consider large Jordan block matrices subject to small Gaussian random perturbations. A result by E.B. Davies and M. Hager [16] shows that as the dimension of the matrix gets large, with probability close to 1, most of the eigenvalues are close to a circle. They, however, only state a logarithmic upper bound on the number of eigenvalues in the interior of that circle.

We study the expected eigenvalue density of the perturbed Jordan block in the interior of that circle and give a precise asymptotic description. Furthermore, we show that the leading contribution of the density is given by the Lebesgue density of the volume form induced by the Poincaré metric on the disc $D(0,1)$.

Keywords Spectral theory; Non-self-adjoint operators; Semiclassical differential operators; Random perturbations.

CONTENTS

INTRODUCTION

The main focus of this thesis lies on the spectral theory of random non-self-adjoint operators. In the case of self-adjoint or more generally normal operators on a complex Hilbert space we have a very good spectral theory due to the spectral theorem. However, for non-normal operators there is no such general result. This produces new challenges and makes the approach to this theory quite varied and exciting. Studying non-self-adjoint problems is an important area of mathematical research as they appear naturally in many different problems, such as

- in the theory of linear partial differential equations given by non-normal operators, e.g.:
  - the solvability theory
  - evolution equations given by a non-normal operator
  - the Kramers-Fokker-Planck type operators
  - the damped wave equation
  - linearized operators from models in fluid dynamics
- in mathematical physics, for example when studying scattering poles, also known as quantum resonances.

We begin by recalling some basic facts from operator theory. Let $\mathscr{H}$ be a separable complex Hilbert space and let $P : D(P) \to \mathscr{H}$ be a closed linear operator with domain $D(P)$, dense in $\mathscr{H}$. We denote the resolvent set of P by

$$\rho(P) := \left\{z \in \mathbb{C};\ (P-z) : D(P) \to \mathscr{H} \text{ is bijective with bounded inverse}\right\}.$$

For $z \in \rho(P)$ we call $(P-z)^{-1}$ the resolvent of P at z. The spectrum of P is defined as

$$\sigma(P) := \mathbb{C} \backslash \rho(P).$$

To define the adjoint of P, set

$$D(P^*) := \{u \in \mathscr{H};\ \exists v \in \mathscr{H} :\ (Pw|u) = (w|v) \text{ for all } w \in D(P)\}.$$

For each such $u \in D(P^*)$, we define $P^* u = v$ where P^* is called the adjoint of P. If $P^* = P$, we say that P is self-adjoint. In this case (and even more generally in the case of normal operators, that is when $P^*P = PP^*$) the spectral theorem (cf for example [56]) yields the following resolvent bound:

$$\|(P-z)^{-1}\| = \frac{1}{\operatorname{dist}(z,\sigma(P))}.$$

A striking difference to the case of normal operators is that when dealing with a non-normal operators $P : D(P) \to \mathscr{H}$ the norm of the resolvent of P can be very large even far away from the spectrum $\sigma(P)$, as generally we only have the lower bound

$$\|(P-z)^{-1}\| \geq \frac{1}{\operatorname{dist}(z,\sigma(P))}.$$

Consequently, the spectrum can be highly unstable even under very small perturbations. This can be profoundly troublesome, for example, in numerical mathematics when we are interested in calculating the eigenvalues of a large non-normal matrix. It can, however, be also the source of many interesting effects.

Spectral instability and pseudospectrum Interest in the phenomenon spectral instability has sparked renewed activity in the study of non-self-adjoint operators originating in numerical analysis. It has been studied, amongst others, by L.N. Trefethen in [80] where he was interested in computing numerically the eigenvalues of large non-normal matrices. Such matrices can come for example from discretizations of differential operators. Understanding spectral instability is in this case of vital importance for the precision of the numerical result. Emphasized by the works of L.N. Trefethen, M. Embree, E.B. Davies, M. Zworski, J. Sjöstrand, cf. [20, 80, 13, 12, 14, 17, 67, 51, 53, 9], and many others, spectral instability of non-self-adjoint operators, in particular the case of (pseudo-)differential operators has been become an active area of research.

A crucial tool for quantifying the spectral instability is the ε-pseudospectrum which, in addition to the spectrum, consists of the superlevel sets of the resolvent, i.e. the points in the resolvent set where the norm of the resolvent is larger than $1/\varepsilon$. Following L.N. Trefethen and M. Embree [20], it can be defined as follows.

Definition 0.0.1. Let P be a closed linear operator on a Hilbert space $\mathscr{H}$ and let $\varepsilon > 0$ be arbitrary. We denote the set of bounded operators on $\mathscr{H}$ by $\mathscr{B}(\mathscr{H})$. Then, $\sigma_\varepsilon(P)$, the ε-*pseudospectrum* of P is defined by

$$\sigma_\varepsilon(P) := \left\{z \in \rho(P);\ \|(P-z)^{-1}\| > \varepsilon^{-1}\right\} \cup \sigma(P), \tag{0.0.1}$$

or equivalently

$$\sigma_\varepsilon(P) = \bigcup_{\substack{B \in \mathscr{B}(\mathscr{H}) \\ \|B\| < \varepsilon}} \sigma(P+B), \tag{0.0.2}$$

or equivalently

$$z \in \sigma_\varepsilon(P) \quad \Longleftrightarrow \quad z \in \sigma(P) \text{ or } \exists u \in D(P),\ \|u\| = 1 \text{ s.t.: } \|(P-z)u\| < \varepsilon. \tag{0.0.3}$$

The last condition also implicitly defines the so-called *quasimodes* or ε-*pseudoeigenvectors*.

Example: Large Jordan block Let us consider the example of a large Jordan block A_0 :

$$A_0 = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix} : \mathbb{C}^N \to \mathbb{C}^N.$$

A_0 is clearly non-normal and has the spectrum $\sigma(A_0) = \{0\}$. Perturbations of a large Jordan block have already been studied, cf. [74, 86, 16, 30]. We will discuss the contributions of these authors in more detail further on in this text. M. Zworski [86] noticed that for every $z \in D(0,1)$, there are associated exponentially accurate quasimodes when $N \to \infty$. Hence the open unit disc is a region of spectral instability.

A simple way to see this is to notice that the Jordan block A_0 is nil-potent, i.e. $A_0^N = 0$. Therefore, for $0 < |z| < 1$ using a Neumann series, one computes that

$$(A_0 - z)^{-1} = -\frac{1}{z}\sum_{n=0}^{N-1} (-z^{-1}A_0)^n.$$

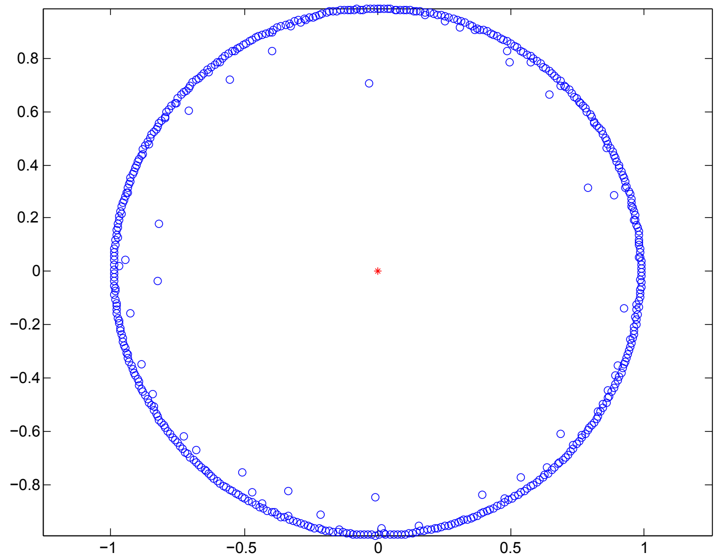

Figure 1: The red star (in the center) depicts the spectrum of A_0 and the blue circles show the eigenvalues of A_δ, a perturbation of A_0 ($N = 500$) with a Gaussian random matrix and coupling constant $\delta = 10^{-4}$.

Setting $e_N = (0,\dots,0,1)^t \in \mathbb{C}^N$, it follows that

$$\|(A_0 - z)^{-1}\| \geq \|(A_0 - z)^{-1} e_N\| \geq \frac{1}{|z|^N},$$

where we use the matrix norm corresponding to the 2-norm on $\mathbb{C}^N$. For $0 < |z| < 1$ the norm of the resolvent of A_0 is much larger than the inverse of the distance of z to the spectrum of A_0 (drastically opposed to what we would expect in the self-adjoint case), since here

$$\|(A_0 - z)^{-1}\| \gg \frac{1}{\operatorname{dist}(z, \sigma(A_0))} = \frac{1}{|z|}.$$

In other words, the disc $|z| < \eta < 1$ is contained in the η^N-pseudospectrum of A_0.

In $\mathbb{C} \setminus \overline{D(0,1)}$ we have spectral stability (a good resolvent estimate), since $\|A_0\| = 1$ which implies that for $|z| > 1$

$$\|(A_0 - z)^{-1}\| \leq \frac{1}{|z| - 1}.$$

Thus, if $A_\delta = A_0 + \delta Q$ is a small perturbation of A_0 we expect the eigenvalues to move inside a small neighborhood of $\overline{D(0,1)}$ (cf Figure 1). In the special case when $Qu = (u|e_1)e_N$, where $(e_j)_1^N$ is the canonical basis in $\mathbb{C}^N$, the eigenvalues of A_δ are of the form

$$\delta^{1/N} e^{2\pi i k/N}, \; k \in \mathbb{Z}/N\mathbb{Z},$$

so if we fix $0 < \delta \ll 1$ and let $N \to \infty$, the spectrum "will converge to a uniform distribution on S^1".

Example: Evolution equations Consider the case of evolution equations given by non-normal operator: Let

$$\begin{cases} \partial_t u(t,x) = Pu(t,x), \\ u(x,0) = u_0(x), \end{cases} \tag{0.0.4}$$

where we suppose that P is a closed, non-normal, densely defined operator on some complex Hilbert space $\mathscr{H}$. A solution to (0.0.4) is formally given by $\mathrm{e}^{tP}u_0(x)$. However, for this expression to make sense, we need to know when P is the generator of a semi-group. The Hille-Yosida theorem (cf [21, 84]) states that P is the generator of e^{tP} $(t \geq 0)$, a contraction semi-group (i.e. $\|\mathrm{e}^{tP}\| \leq 1$) if and only if

$$]0,\infty[\subset \rho(P) \quad \text{and} \quad \|(P-\lambda)^{-1}\| \leq \lambda^{-1} \text{ for } \lambda > 0.$$

On the other hand we have the following lower bound on the semi-group

$$\|e^{tP}\| \geq e^{\gamma t},\ \forall t \geq 0, \quad \text{where} \quad \gamma = \sup_{z \in \sigma(P)} \operatorname{Re} z.$$

The precision of this bound depends strongly on the spectrum of P and is therefore strongly influenced by the effects of spectral instability. This can become of particular relevance when we are interested in solutions with respect to small perturbations of P or for the stability of numerical algorithms.

In the case of a certain class of non-linear evolution equation B. Sandstede and A. Scheel [58] showed that in spite of the problem being spectrally stable (meaning that the relevant linearized operator has its spectrum in $\operatorname{Re} z < \gamma < 0$) the solutions blow up with arbitrarily small initial data. This was generalized by J. Galkowski [24, 25] to a large class of non-linear evolution problems. He linked the blow up of the solutions to the fact that although the spectrum of the linearized problem is uniformly bounded away from $\operatorname{Re} z \geq 0$, the pseudospectrum of this operator has non-empty intersection with $\operatorname{Re} z \geq 0$. He emphasized therefore the importance of the pseudospectrum for the study of stability of solutions to non-linear evolution equations.

For a similar and simpler example illustrating with a Jordan block matrix this pseudospectral instability for non-linear systems, we refer the reader to the work of A. Raphael and M. Zworski [54, Sec. 3].

Example: Resonances Questions regarding the spectral theory of non-self-adjoint operators can appear very naturally even when studying a self-adjoint problem to begin with. A prominent example for this is the study of scattering poles or resonances for the Schrödinger equation in mathematical physics.

Recall that a particle such as an electron immersed in an electrostatic potential (as in the case of a hydrogen atom where the electron is immersed in the electrostatic potential emitted by a proton) moving through d-dimensional space is described via a square integrable function $\psi_0 \in L^2(\mathbb{R}^d)$ called a state. According to the Copenhagen interpretation of quantum mechanics the quantity

$$\left(\int_A |\psi_0|^2 dx\right)^{\frac{1}{2}}, \quad A \subset \mathbb{R}^d \text{ measurable},$$

corresponds to the probability to find the particle in A. The time evolution ($t \geq 0$) of the state ψ_0 is determined by the Schrödinger equation

$$\begin{cases} i\partial_t \psi(t,x) = H\psi(t,x), \\ \psi(0,x) = \psi_0(x). \end{cases} \tag{0.0.5}$$

Here $H = -\Delta + V$ is called the Schrödinger operator where Δ denotes the Laplace operator and V a multiplication operator describing an electrostatic potential. We assume here $V \in L^\infty_{comp}(\mathbb{R}^n;\mathbb{R})$ for simplicity. H is an unbounded self-adjoint operator in $L^2(\mathbb{R}^n)$ with domain given by the Sobolev space $H^2(\mathbb{R}^n)$. The essential spectrum of H is given by $[0,+\infty[$ (i.e. the essential spectrum of $-\Delta$) and in $]-\infty,0[$ there can be only discrete eigenvalues $-\mu_j^2$ which correspond to bounded states of the system determined by H.

The equations (0.0.5) have a unique solution given by $e^{-itH}\psi_0$. For the large time evolution, we need to take into account not only the effects of the discrete spectrum but also of the essential spectrum. A way to do this is by considering resonances which are given by showing that the resolvent $(H-\lambda^2)^{-1}$ has a meromorphic continuation (cf [65]) from the upper half plane $\mathbb{C}^+$ to

- $\mathbb{C}$, in case the dimension n is odd,
- the logarithmic covering space of $\mathbb{C}^*$, in case the dimension n is even,

with values in the bounded operators from $H^0_{comp}(\mathbb{R}^n)$ to $H^2_{loc}(\mathbb{R}^n)$. The poles of this meromorphic continuation are called resonances, with exception of the $i\mu_j$, and they can be used to study the large time behavior of e^{-iHt}, in particular to expand solutions to the Schrödinger equation in exponentially decaying resonant modes. It is the resonances closest to real axis that give the principal contribution to this, wherefore there has been a large interest in the studying those for various operators (see for example [72, 73, 6, 7, 43, 77, 78, 45, 70]).

However, finding the poles of the meromorphic continuation is not a self-adjoint problem anymore. Therefore, effects from spectral instability become relevant and interesting, as for example in the case of resonances of Random Schrödinger equations where we consider equations of the same type as (0.0.5) with the potential V being random. This describes physical systems of particles being immersed in a random environment which can be used to model for example disordered system such as “dirty” (super-)conductors, see for example [8, 42].

Objective - Random perturbations

In view of (0.0.2) it is very natural to investigate the effects of small random perturbations upon the spectrum of non-self-adjoint operators. The principal aim of this thesis is to study this in the following two cases:

Semiclassical differential operators We will study the effects of small random perturbations on the spectrum of non-normal semiclassical differential operators. Our principal interest is to study the average density of eigenvalues and their two-point interaction. We will discuss the framework, previous results and new results obtained in this thesis in Sections 1.1, 1.2 and 1.3.

Jordan block matrices We will consider Jordan block matrices subject to small random perturbations and study the average density of eigenvalues in the interior of the zone of spectral instability (as described above). We will discuss the previous results and obtained results in Section 1.4.

Organization of this thesis

Before continuing we give a short overview on the structure of this thesis.

Chapter 1 In the first chapter we present an introduction to some problems and questions concerning the spectral instability of non-self-adjoint operators. We will also discuss the specific framework of the two principal problems under consideration in this thesis, that is small random perturbations of a class of non-self-adjoint semiclassical differential operators and of large Jordan matrices. As mentioned above the results obtained in this work concerning both cases will be discussed in Sections 1.2, 1.3 and 1.4.

Chapter 2 In this chapter we will prove the results discussed in Section 1.2. We will present constructions of quasimodes, Grushin problems and techniques developed in this thesis to obtain results on the average density of eigenvalues of a certain class of non-self-adjoint semiclassical differential operators.

Chapter 3 In this chapter we will continue to treat the class of operators under consideration in Chapter 2 and present the proofs of the results discussed in Section 1.3. We will build on the methods displayed in Chapter 2.

Chapter 4 This chapter deals with the case of random perturbations of large Jordan matrices. We discuss the proofs of the results presented in Section 1.4. The techniques used in this chapter are similar to those of Chapter 2.

Appendix A In the appendix will display the MATLAB code used to obtain the numerical simulations presented throughout this thesis.

Notation

In this work we are going to use the following notations:

1. We will denote the Lebesgue measure on $\mathbb{C}^d$ by $L(dz)$.
2. For $\alpha \in \mathbb{N}^d$, we define $|\alpha| := |\alpha_1| + \cdots + |\alpha_d|$ and in particular, for $d = 3$, we set
$$\partial^{\alpha}_{z\overline{z}x} := \partial^{\alpha_1}_{z}\partial^{\alpha_2}_{\overline{z}}\partial^{\alpha_3}_{x}.$$
3. We denote by $f(x) \asymp g(x)$ that there exists a constant $C > 0$ such that
$$C^{-1}g(x) \leq f(x) \leq Cg(x).$$
Moreover, when we write $\asymp \eta$, we mean some function f such that $f \asymp \eta$.
4. We work with the convention that when we write $f = \mathcal{O}(1)^{-1}$ then we mean implicitly that $0 < f = \mathcal{O}(1)$.
5. We denote by $f(x) \ll g(x)$ that there exists some large constant $C > 1$ such that
$$f(x) \leq C^{-1}g(x).$$
6. We write $\chi_1(x) \succ \chi_2(x)$, with $\chi_i \in \mathcal{C}^{\infty}_0$, if $\operatorname{supp}\chi_2 \subset \complement\operatorname{supp}(1-\chi_1)$.

CHAPTER 1

SPECTRA OF NON-SELF-ADJOINT RANDOM OPERATORS

1.1 | Random perturbations of non-self-adjoint semiclassical differential operators

Semiclassical differential operators and the Weyl law We begin by recalling some standard notions of the framework of semiclassical differential operators which can be found for example in [87, 18].

For $h \in]0,1]$ consider

$$P(x,hD_x) = \sum_{|\alpha|\leq N} a_\alpha(x)(hD_x)^\alpha, \quad D_x = \frac{1}{i}\frac{\partial}{\partial x}, \tag{1.1.1}$$

where $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + \dots + \alpha_n$ and

$$a_\alpha(x) \in \mathscr{C}_b^\infty(\mathbb{R}^n) := \left\{u \in \mathscr{C}^\infty(\mathbb{R}^n);\ \forall \alpha \in \mathbb{N}^n\ \partial^\alpha u \in L^\infty(\mathbb{R}^n).\right\}$$

The natural domain of $P(x,hD_x)$ is the semiclassical Sobolev space $H^N_{sc}(\mathbb{R}^n)$ defined by

$$H^N_{sc}(\mathbb{R}^n) := \left\{ u \in L^2(\mathbb{R}^n);\ \sum_{|\alpha|\leq N} \|(hD_x)^\alpha u\|^2 < \infty \right\}.$$

The formal adjoint of P is given by

$$P^*(x,hD_x) = \sum_{|\alpha|\leq N} (hD_x)^\alpha \overline{a}_\alpha(x).$$

On the phase space $T^*\mathbb{R}^n$, we denote by

$$p(x,\xi) = \sum_{|\alpha|\leq N} a_\alpha(x)\xi^\alpha, \quad (x,\xi) \in T^*\mathbb{R}^n, \tag{1.1.2}$$

the semiclassical principal symbol of $P(x,hD_x)$, and we recall that the Poisson bracket of p and $\overline{p}$ is given by

$$\{p,\overline{p}\} = \partial_\xi p \cdot \partial_x \overline{p} - \partial_x p \cdot \partial_\xi \overline{p}.$$

We are interested in the spectral properties of $P(x,hD_x)$ in the limit $h \to 0$, which is called the semiclassical limit. The fundamental motivation behind studying such limits is to understand the relation between classical dynamics in phase space and quantum mechanics, when $h \to 0$. A famous example is the *Weyl law* for the eigenvalues of the Schrödinger operator

$$P(x, hD_x) = -h^2\Delta + V(x).$$

with a smooth potential $V \in \mathscr{C}^\infty(\mathbb{R}^n;\mathbb{R})$ satisfying suitable growth conditions

$$\begin{cases} |\partial^\alpha V(x)| \leq C_\alpha \langle x\rangle^k, \ \forall \alpha \in \mathbb{N}^n, \\ V(x) \geq c\langle x\rangle^k, \text{ for } |x| \geq R, \end{cases}$$

where $R, k, C_\alpha, c > 0$ are some constants. Then we have the following celebrated result linking the asymptotic behavior (as $h \to 0$) of the number of eigenvalues of $P(x, hD_x)$ in an interval $I \subset \mathbb{R}$ to the symplectic volume in phase space of $p^{-1}(I)$ (which is a classical quantity) where $p = |\xi|^2 + V(x)$ is the semiclassical principal symbol of $P(x, hD_x)$:

Theorem 1.1.1 (*Weyl's law,* see e.g. [87])**.** *Let $P = P(x, hD_x)$ be as above and let $I \subset \mathbb{R}$ be an interval. Then*

$$\#(\sigma(P) \cap I) = \frac{1}{(2\pi h)^n}\left(\iint_{p^{-1}(I)} dx d\xi + o(1)\right).$$

Such a Weyl law is known to hold for a large class of semiclassical self-adjoint pseudo-differential elliptic operators, see for example [18, 39, 34].

Spectral instability for non-self-adjoint semiclassical differential operators Next, we are going to put the concept of spectral instability into context with the framework of semiclassical differential operators.

Since the principal symbol of the commutator

$$\frac{1}{h}[P, P^*] = \frac{1}{h}(PP^* - P^*P)$$

is given by $i^{-1}\{p, \overline{p}\}$, we see that the Poisson bracket of p and $\overline{p}$ being different than zero implies that the operator P is non-normal.

In [15] E.B. Davies considers the one dimensional Schrödinger operator with complex potential

$$P(x, hD_x) = (hD_x)^2 + V(x), \quad V \in \mathscr{C}^\infty(\mathbb{R}) \tag{1.1.3}$$

and gives a construction of quasimodes. He proves that for all $(x,\xi) \in T^*\mathbb{R}$ satisfying $\xi \neq 0$ and $\operatorname{Im} V'(x) \neq 0$, and all $N \in \mathbb{N}$

$$\exists u_h \in L^2(\mathbb{R}), \ \|(P(x, hD_x) - z)u_h\| \leq C_N h^N \|u_h\|_{L^2}, \quad z = \xi^2 + V(x).$$

K. Pravda-Starov generalized this in [51] by observing that there also exist quasimodes corresponding to points $(x,\xi) \in T^*\mathbb{R}$ satisfying

$$\xi \neq 0, \ \operatorname{Im} V^{(j)}(x) = 0, \text{ for } j = 1, \ldots, 2p, \text{ and } \operatorname{Im} V^{(2p+1)}(x) \neq 0.$$

M. Zworski then observed in [85] a relation between Davies' quasimode construction and L. Hörmander's Poisson bracket condition (cf. [36]) in the context of local non-solvability of linear partial differential equations stating that a (classical) differential operator $P(x, D_x)$ with smooth coefficients and principal symbol p is non-solvable in an open set $\Omega \subset \mathbb{R}^n$ if

$$\exists \rho \in T^*\Omega\backslash\{0\} \text{ s.t.: } p(\rho) = 0 \text{ and } \frac{1}{2i}\{p, \overline{p}\}(\rho) \neq 0.$$

M. Zworski concluded in [85] from the results of L. Hörmander [37, Sect. 26] and of J.J. Duistermaat and J. Sjöstrand [19] that for P as in (1.1.1) (in fact for the more general case of semiclassical pseudo-differential operators) with semiclassical principal symbol p (see (1.1.2)) we have that for all

$$z \in \Lambda_-(p) = \left\{p(\rho) : \left\{\mathrm{Re}\, p, \mathrm{Im}\, p\right\}(\rho) < 0\right\} \subset \Sigma(p) := \overline{p(T^*\mathbb{R}^n)} \tag{1.1.4}$$

there exists a $u_h \in L^2(\mathbb{R}^n)$ with the property

$$\|(P(x,hD_x) - z)u_h\| = \mathscr{O}\left(h^\infty\right)\|u_h\|_{L^2}, \tag{1.1.5}$$

where u_h is localized to a point in phase space ρ with $p(\rho) = z$, i.e. $\mathrm{WF}_h(u_h) = \{\rho\}$. We recall that for $v = v(h)$, $\|v\|_{L^2} = \mathscr{O}(h^{-N})$, for some fixed N, the semiclassical wavefront set of v, $\mathrm{WF}_h(v)$, is defined by

$$\complement\left\{(x,\xi) \in T^*\mathbb{R}^n : \exists a \in \mathscr{S}(T^*\mathbb{R}^n),\ a(x,\xi) = 1,\ \|a^w v\|_{L^2} = \mathscr{O}(h^\infty)\right\}$$

where a^w denotes the Weyl quantization of a, i.e.

$$a^w(x,hD_x)v(x) := \frac{1}{(2\pi h)^n}\iint \mathrm{e}^{\frac{i}{h}(x-y)\cdot\eta}\, a\left(\frac{x+y}{2},\eta\right) v(y)\,dy\,d\eta.$$

In the case where $P(x,hD_x)$ has analytic coefficients, we may replace $\mathscr{O}(h^\infty)$ with $\mathscr{O}(\mathrm{e}^{-1/Ch})$ in (1.1.5). We also refer the reader to the thesis of K. Pravda-Starov [52] where he relates the non-negativity of odd iterations of the above bracket condition to the construction of quasimodes similar to the above.

In case of the one dimensional semiclassical Schrödinger operator with complex potential (1.1.3) considered by E.B. Davies, the condition $\left\{\mathrm{Re}\, p, \mathrm{Im}\, p\right\}(x,\xi) < 0$ from (1.1.4) simplifies to $\mathrm{Im}\, V'(x) \neq 0$ and $\xi \neq 0$, as shown by Davies.

In the case of multi-dimensional semiclassical Schrödinger operator with smooth complex potential the bracket condition from (1.1.4) becomes $\mathrm{Im}\,(\xi|\partial_x V(x)) \neq 0$.

Finally, let us remark that N. Dencker, J. Sjöstrand and M. Zworski give in [17] a direct proof of (1.1.5) (also in the context of semiclassical pseudo-differential operators).

1.1.1 – Hager's model

To study the effects of spectral instability in the framework of semiclassical (pseudo-)differential operators, M. Hager introduced in [32] the following model operator:

Hypothesis 1.1.2 (Hager's model). Let $0 < h \ll 1$, we consider on $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ the semiclassical differential operator $P_h : L^2(S^1) \to L^2(S^1)$ given by

$$P_h := hD_x + g(x), \quad D_x := \frac{1}{i}\frac{d}{dx}, \quad g \in \mathscr{C}^\infty(S^1), \tag{1.1.6}$$

where $g \in \mathscr{C}^\infty(S^1)$ is such that $\mathrm{Im}\, g$ has exactly two critical points and they are non-degenerate, one minimum and one maximum, say in a and b, with $a < b < a + 2\pi$ and $\mathrm{Im}\, g(a) < \mathrm{Im}\, g(b)$. Without loss of generality we may assume that $\mathrm{Im}\, g(a) = 0$.

The natural domain of P_h is the semiclassical Sobolev space

$$H^1_{sc}(S^1) := \left\{u \in L^2(S^1) : \left(\|u\|^2 + \|hD_x u\|^2\right)^{\frac{1}{2}} < \infty\right\},$$

where $\|\cdot\|$ denotes the L^2-norm on S^1 if nothing else is specified. We will use the standard scalar products on $L^2(S^1)$ and $\mathbb{C}^N$ defined by

$$(f|g) := \int_{S^1} f(x)\overline{g}(x)\,dx, \quad f,g \in L^2(S^1),$$

and

$$(X|Y) := \sum_{i=1}^{N} X_i \overline{Y}_i, \quad X,Y \in \mathbb{C}^N.$$

We denote the semiclassical principal symbol of P_h by

$$p(x,\xi) = \xi + g(x), \quad (x,\xi) \in T^* S^1. \tag{1.1.7}$$

The spectrum of P_h is discrete with simple eigenvalues, given by

$$\sigma(P_h) = \left\{ z \in \mathbb{C} : \ z = \langle g \rangle + kh, \ k \in \mathbb{Z} \right\}, \tag{1.1.8}$$

where $\langle g \rangle := (2\pi)^{-1} \int_{S^1} g(y) dy$.

1.1.2 – Adding a random perturbation

We are interested in the following random perturbation of P_h:

Hypothesis 1.1.3 (**Random Perturbation of Hager's model**)**.** Let P_h be as in in Hypothesis 1.1.2. Define

$$P_h^\delta := P_h + \delta Q_\omega = hD_x + g(x) + \delta Q_\omega, \tag{1.1.9}$$

where $0 < \delta \ll 1$ and Q_ω is an integral operator $L^2(S^1) \to L^2(S^1)$ of the form

$$Q_\omega u(x) := \sum_{|j|,|k| \leq \left\lfloor \frac{C_1}{h} \right\rfloor} \alpha_{j,k} (u|e^k) e^j(x). \tag{1.1.10}$$

Here $\lfloor x \rfloor := \max\{n \in \mathbb{N} : \ x \geq n\}$ for $x \in \mathbb{R}$, $C_1 > 0$ is big enough, $e^k(x) := (2\pi)^{-1/2} \mathrm{e}^{ikx}$, $k \in \mathbb{Z}$, and $\alpha_{j,k}$ are complex valued independent and identically distributed random variables with complex Gaussian distribution law $\mathscr{N}_{\mathbb{C}}(0,1)$.

Recall that a random variable α has complex Gaussian distribution law $\mathscr{N}_{\mathbb{C}}(0,1)$ if

$$\alpha_*(\mathbb{P}(d\omega)) = \frac{1}{\pi} \mathrm{e}^{-\alpha\overline{\alpha}} L(d\alpha)$$

where $L(d\alpha)$ denotes the Lebesgue measure on $\mathbb{C}$ and ω is the random parameter living in the sample space $\mathscr{M}$ of a probability space $(\mathscr{M}, \mathscr{A}, \mathbb{P})$ with σ-algebra $\mathscr{A}$ and probability measure $\mathbb{P}$. $\alpha \sim \mathscr{N}_{\mathbb{C}}(0,1)$ implies that

$$\mathbb{E}[\alpha] = 0, \quad \text{and} \quad \mathbb{E}\left[|\alpha|^2\right] = 1,$$

or in other words $\alpha \sim \mathscr{N}_{\mathbb{C}}(0,1)$ has expectation 0 and variance 1. In the above, $\mathbb{E}[\cdot]$ denotes the expectation with respect to the random variables.

The following results were obtained by W. Bordeaux-Montrieux [4].

Proposition 1.1.4 (W. Bordeaux-Montrieux [4])**.** *There exists a $C_0 > 0$ such that the following holds: Let $X_j \sim \mathscr{N}_{\mathbb{C}}(0, \sigma_j^2)$, $1 \leq j \leq N < \infty$ be independent complex Gaussian random variables. Put $s_1 = \max \sigma_j^2$. Then, for every $x > 0$, we have*

$$\mathbb{P}\left[\sum_{j=1}^N |X_j|^2 \geq x\right] \leq \exp\left(\frac{C_0}{2s_1} \sum_{j=1}^N \sigma_j^2 - \frac{x}{2s_1}\right).$$

Corollary 1.1.5 (W. Bordeaux-Montrieux [4])**.** *Let $h > 0$ and let $\|Q_\omega\|_{\mathrm{HS}}$ denote the Hilbert-Schmidt norm of Q_ω. If $C > 0$ is large enough, then*

$$\|Q_\omega\|_{HS} \leq \frac{C}{h} \quad \textit{with probability} \geq 1 - \mathrm{e}^{-\frac{1}{Ch^2}}.$$

Here, the constant $C > 0$ in the probability estimate is not necessarily the same as before.

Since $\|Q_\omega\|_{HS}^2 = \sum |\alpha_{j,k}(\omega)|^2$, we can also view the above bound as restricting the support of the joint probability distribution of the random vector $\alpha = (\alpha_{jk})_{j,k}$ to a ball of radius C/h. Hence, to obtain a bounded perturbation we will work from now on in the restricted probability space:

Hypothesis 1.1.6 (Restriction of random variables). Define $N := (2\lfloor C_1/h\rfloor + 1)^2$ where $C_1 > 0$ is as in (1.1.10). We assume that for some constant $C > 0$

$$\alpha \in B(0,R) \subset \mathbb{C}^N, \quad R = \frac{C}{h}. \tag{1.1.11}$$

Furthermore, we assume that the coupling constant $\delta > 0$ satisfies

$$\delta \ll h^{5/2}, \tag{1.1.12}$$

which implies, for $\alpha \in B(0,R)$, that $\delta \|Q_\omega\|_{HS} \leq Ch^{3/2}$. Hence, for $\alpha \in B(0,R)$, the operator Q_ω is compact and the spectrum of P_h^δ is discrete.

Zone of spectral instability Since in the present work we are in the semiclassical setting, we define similarly to (1.1.4)

$$\Sigma := \overline{p(T^*S^1)} \subset \mathbb{C}, \tag{1.1.13}$$

where p is given in (1.1.7). In the case of (1.1.6) and (1.1.7) $p(T^*S^1)$ is already closed due to the ellipticity of P_h.

Next, consider for $z \in \Omega \Subset \mathring{\Sigma}$ the equation $z = p(x,\xi)$. It has precisely two solutions $\rho_\pm(z) := (x_\pm(z), \xi_\pm(z))$ where $x_\pm(z)$ are given by

$$\operatorname{Im} g(x_\pm(z)) = \operatorname{Im} z, \quad \pm \operatorname{Im} g'(x_\pm(z)) < 0 \quad \text{and} \quad \xi_\pm(z) = \operatorname{Re} z - \operatorname{Re} g(x_\pm(z)). \tag{1.1.14}$$

By the natural projection $\Pi : \mathbb{R} \to S^1 = \mathbb{R}/2\pi\mathbb{Z}$ and a slight abuse of notation we identify the points $x_\pm \in S^1$ with points $x_\pm \in \mathbb{R}$ such that $x_- - 2\pi < x_+ < x_-$. Furthermore, we will identify S^1 with the interval $[x_- - 2\pi, x_-[$.

Therefore, we see that in the case of (1.1.7), the bracket condition given in (1.1.4) is satisfied for any $z \in \Omega \Subset \mathring{\Sigma}$ since by (1.1.14)

$$\left\{\operatorname{Re} p, \operatorname{Im} p\right\}(x_+(z), \xi_+(z)) = \operatorname{Im} g'(x_+(z)) < 0.$$

We will give more details on the construction of quasimodes for P_h in Section 2.1.

For z close to the boundary of Σ the situation is different as we have a good resolvent estimate on $\partial\Sigma$. Since $\{p, \{p, \overline{p}\}\}(\rho) \neq 0$ for all $z_0 \in \partial\Sigma$ and all $\rho \in p^{-1}(z_0)$, Theorem 1.1 in [69] implies that there exists a constant $C_0 > 0$ such that for every constant $C_1 > 0$ there is a constant $C_2 > 0$ such that for $|z - z_0| < C_1(h \ln \frac{1}{h})^{2/3}$, $h < \frac{1}{C_2}$, the resolvent $(P_h - z)^{-1}$ is well defined and satisfies

$$\|(P_h - z)^{-1}\| < C_0 h^{-\frac{2}{3}} \exp\left(\frac{C_0}{h} |z - z_0|^{\frac{3}{2}}\right). \tag{1.1.15}$$

This implies for α as in (1.1.11) and $\delta = \mathcal{O}(h^M)$, $M = M(C_1, C) > 0$ large enough, that

$$\sigma(P_h + \delta Q_\omega) \cap D\left(z_0, C_1 \left(h \ln \frac{1}{h}\right)^{2/3}\right) = \emptyset. \tag{1.1.16}$$

Thus, there exists a tube of radius $C_1 \left(h \ln \frac{1}{h}\right)^{2/3}$ around $\partial\Sigma$ void of the spectrum of the perturbed operator P_h^δ. Therefore, since we are interested in the eigenvalue distribution of P_h^δ, we assume from now on implicitly that

Hypothesis 1.1.7 (Restriction of Σ). Let $\Sigma \subset \mathbb{C}$ be as in (1.1.13). Then, we let

$$\Omega \Subset \Sigma \text{ be open, relatively compact with } \operatorname{dist}(\Omega, \partial\Sigma) > C\left(h \ln h^{-1}\right)^{2/3} \text{ for some constant } C > 0. \tag{1.1.17}$$

1.1.3 – Review of previous and related results

In [32] M. Hager showed the striking result that, although the eigenvalues of P_h (cf (1.1.8)) do not follow a Weyl law, after adding a tiny random perturbation the eigenvalues of the perturbed operator P_h^δ follow in the interior of Σ a Weyl law with probability very close to one:

Theorem 1.1.8 (M. Hager [32])**.** *Let $\Omega \Subset \mathring{\Sigma}$ be open and relatively compact such that* $\operatorname{dist}(\Omega, \partial\Sigma) > 1/C$*, for a $C \gg 1$. Let $\Gamma \Subset \Omega$ be with $\mathscr{C}^\infty$ boundary. Let $\kappa > 5/2$ and let $\varepsilon_0 > 0$ be sufficiently small. Let $\delta = \delta(h)$ satisfy $\mathrm{e}^{-\varepsilon_0/h} \ll \delta \ll h^\kappa$ and put $\varepsilon = \varepsilon(h) = h\ln(1/\delta)$. Then with probability $\geq 1 - \mathscr{O}\left(\frac{\delta^2}{\sqrt{\varepsilon}h^5}\right)$,*

$$\#\left(\sigma(P_h^\delta) \cap \Gamma\right) = \frac{1}{2\pi h}\iint_{p^{-1}(\Gamma)} dx d\xi + \mathscr{O}\left(\frac{\sqrt{\varepsilon}}{h}\right).$$

M. Hager's result is particularly interesting when

$$\frac{\delta^2}{\sqrt{\varepsilon}h^5} \ll 1 \quad \text{and} \quad \sqrt{\varepsilon} \ll 1,$$

as it would be for example the case when

$$\ln\frac{1}{\delta} \ll \frac{1}{h}, \quad \delta \ll h^{\frac{11}{4}}.$$

Hager's result has been extended by W. Bordeaux-Montrieux in [4] to strips at a distance $\gg (-h\ln\delta h)^{\frac{2}{3}}$ to the boundary of Σ:

$$\Gamma_\tau := \{z \in \Sigma;\ C_1 \leq \operatorname{Re} z \leq C_2,\ \operatorname{Im} z \asymp \tau\}, \quad \text{with } (-h\ln(\delta h))^{2/3} \ll \tau \ll 1 \tag{1.1.18}$$

where C_1, C_2 are constants independent of τ. W. Bordeaux-Montrieux showed in [4] that the eigenvalues of the perturbed operator P_h^δ follow also in Γ_τ a Weyl law:

Theorem 1.1.9 (W. Bordeaux-Montrieux [4])**.** *Let $\kappa, \gamma > 0$ and let Γ_τ be as above. Let $\delta = \delta(h)$ satisfy*

$$Ch^\kappa \leq \delta \ll \frac{\sqrt{h}\tau^{1/4}h^{2\gamma}}{(\ln h^{-1})^3},$$

and put $\varepsilon = \varepsilon(h) = C_\gamma h \ln(h\delta)^{-1}$. Then with probability $\geq 1 - \mathscr{O}\left(\frac{h^{2\gamma}}{\sqrt{\varepsilon}\tau^{1/4}}\right)$,

$$\#\left(\sigma(P_h^\delta) \cap \Gamma_\tau\right) = \frac{1}{2\pi h}\iint_{p^{-1}(\Gamma_\tau)} dx d\xi + \mathscr{O}\left(\frac{\sqrt{\varepsilon}}{\tau^{1/4}h}\right).$$

Furthermore, Hager and Bordeaux-Montrieux generalized their respective results to the case of one-dimensional semiclassical pseudo-differential operators, see [31, 4]. In [33], M. Hager and J. Sjöstrand generalized Hager's result to the case of multi-dimensional semiclassical pseudo-differential operators.

There are many more interesting results about Weyl asymptotics of the eigenvalues of non-self-adjoint operators: in [9] M. Zworski and T.J. Christiansen proved a probabilistic Weyl law for the eigenvalues in the setting of small random perturbations of Toeplitz quantizations of complex-valued functions on an even dimensional torus. In [68, 66] J. Sjöstrand proved a Weyl law for the eigenvalues the case small multiplicative random perturbations of multi-dimensional semiclassical pseudo-differential operators similar to the class under consideration in [33].

1.1.4 – Questions treated

The above mentioned results concern only the eigenvalues in the interior of the pseudospectrum and numerical simulations suggest that Weyl asymptotics break down when we approach the boundary of the pseudospectrum (cf Figures 1.1, 1.2 and 1.3). Furthermore, there have not been any results concerning the statistical interaction between eigenvalues.

Therefore, in the first part of this thesis we go back to the model operator P_h introduced by Hager (cf Hypothesis 1.1.2) and we are interested in the following questions:

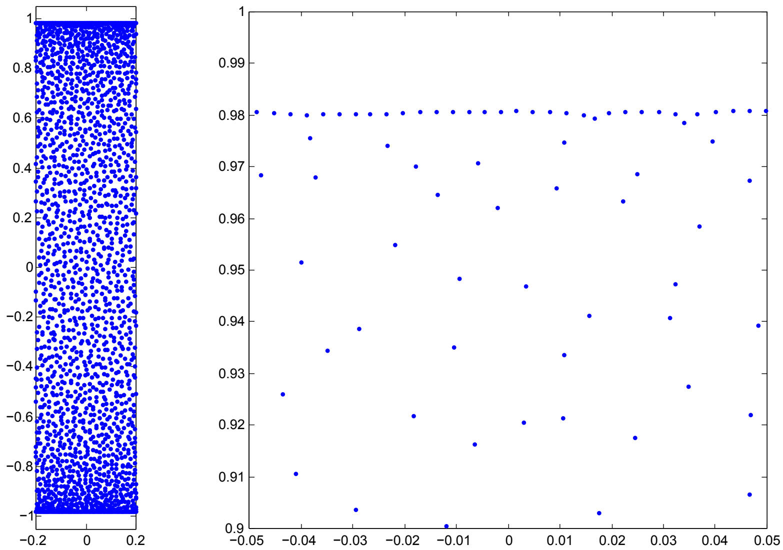

Figure 1.1: Sections of the spectrum of the discretization of $hD + \exp(-ix)$ (approximated by a 6000×6000-matrix) perturbed with a random Gaussian matrix δR with $h = 2 \cdot 10^{-4}$ and $\delta = 2 \cdot 10^{-14}$. The right hand side is a magnification of the upper boundary region of the picture on the left hand side.

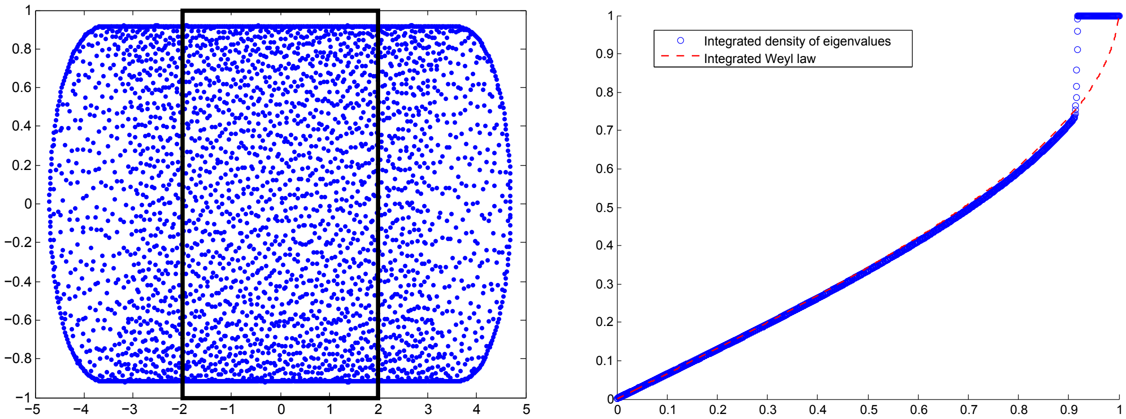

Figure 1.2: On the left hand side we present the spectrum of the discretization of $hD + \exp(-ix)$ (approximated by a 3999×3999-matrix) perturbed with a random Gaussian matrix δR with $h = 2 \cdot 10^{-3}$ and $\delta = 2 \cdot 10^{-12}$. The black box indicates the region where we count the number of eigenvalues to obtain the image on the right hand side. There we show the integrated experimental density of eigenvalues, averaged over 400 realizations of random Gaussian matrices, and the integrated Weyl law. We can see clearly a region close to the boundary of the pseudospectrum where Weyl asymptotics of the eigenvalues breaks down.

1) Density of eigenvalues What is the precises description of the density of eigenvalues of the randomly perturbed operator P_h^δ (cf (1.1.9)) in all of Σ (cf (1.1.13))?

2) 2**-point interaction of eigenvalues** How is the two-point interaction of eigenvalues of P_h^δ in the interior of Σ? Is it repulsive, attractive or neither?

1.2 | Average density of eigenvalues of Hager's model

We begin by establishing how to choose the strength of the perturbation. For this purpose we discuss some estimates on the norm of the resolvent of P_h.

1.2.1 – The coupling δ

We give a description of the imaginary part of the action between $\rho_+(z)$ and $\rho_-(z)$.

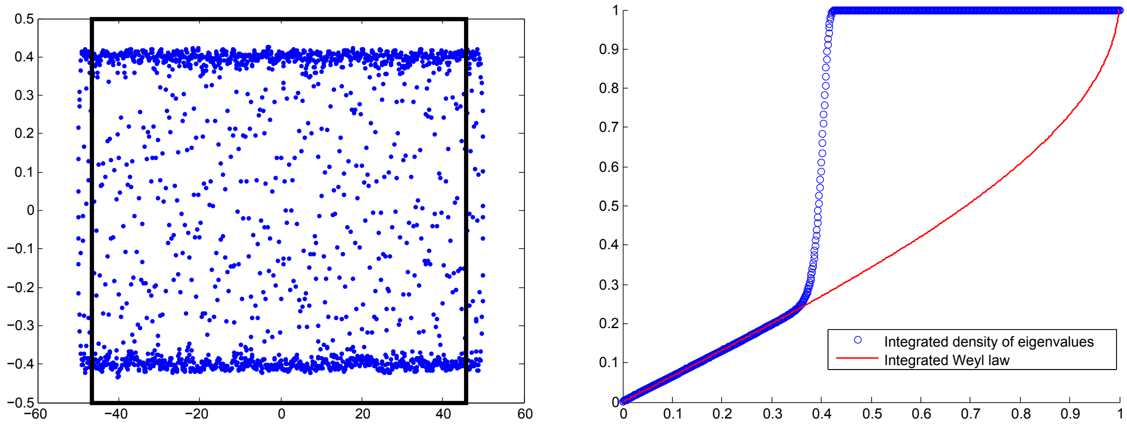

Figure 1.3: On the left hand side we present the spectrum of the discretization of $hD+\exp(-ix)$ (approximated by a 1999×1999-matrix) perturbed with a random Gaussian matrix δR with $h=5\cdot 10^{-2}$ and $\delta=\exp(-1/h)$. The black box indicates the region where we count the number of eigenvalues to obtain the image on the right hand side. There we show the integrated experimental density of eigenvalues, averaged over 400 realizations of random Gaussian matrices, and the integrated Weyl law. Here, the Weyl law breaks down even more dramatically than in Figure 1.2.

Remark 1.2.1. Much of the following is valid for $z\in\Omega\Subset\Sigma$ with

$$\Omega\Subset\Sigma \text{ open, relatively compact with } \operatorname{dist}(\Omega,\partial\Sigma)\gg h^{2/3}, \tag{1.2.1}$$

instead of for $z\in\Omega$ as in Hypothesis 1.1.7.

Definition 1.2.2. Let $\Omega\Subset\Sigma$ as in (1.2.1), let p denote the semiclassical principal symbol of P_h in (1.1.7) and let $\rho_\pm(z)=(x_\pm(z),\xi_\pm(z))$ be as above. Define

$$S(z):=\min\left(\operatorname{Im}\int_{x_+}^{x_-}(z-g(y))dy,\operatorname{Im}\int_{x_+}^{x_--2\pi}(z-g(y))dy\right).$$

Proposition 1.2.3. *Let $\Omega\Subset\Sigma$ be as in (1.2.1) and let $S(z)$ be as in Definition 1.2.2, then $S(z)$ has the following properties for all $z\in\Omega$:*

- *$S(z)$ depends only on $\operatorname{Im} z$, is continuous and has the zeros $S(\operatorname{Im} g(a))=S(\operatorname{Im} g(b))=0$;*
- *$S(z)\geq 0$;*
- *for $\operatorname{Im} z=\langle\operatorname{Im} g\rangle$ the two integrals defining S are equal; S has its maximum at $\langle\operatorname{Im} g\rangle$ and is strictly monotonously decreasing on the interval $[\langle\operatorname{Im} g\rangle,\operatorname{Im} g(b)]$ and strictly monotonously increasing on $[\operatorname{Im} g(a),\langle\operatorname{Im} g\rangle]$;*
- *its derivative is piecewise of class $\mathscr{C}^\infty$ with the only discontinuity at $\operatorname{Im} z=\langle\operatorname{Im} g\rangle$. Moreover,*

$$S(z)=\int_{\langle\operatorname{Im} g\rangle}^{\operatorname{Im} z}(\partial_{\operatorname{Im} z}S)(t)dt+S(\langle\operatorname{Im} g\rangle),$$
$$(\partial_{\operatorname{Im} z}S)(t):=\begin{cases}x_-(t)-x_+(t), & \textit{if } t\leq\langle\operatorname{Im} g\rangle,\\ x_-(t)-2\pi-x_+(t), & \textit{if } t>\langle\operatorname{Im} g\rangle.\end{cases} \tag{1.2.2}$$

- *S has the following asymptotic behavior for $z\in\Omega$*

$$S(z)\asymp d(z)^{\frac{3}{2}},\quad \textit{and}\quad |\partial_{\operatorname{Im} z}S(z)|\asymp d(z)^{\frac{1}{2}},$$

where $d(z):=\operatorname{dist}(z,\partial\Sigma)$.

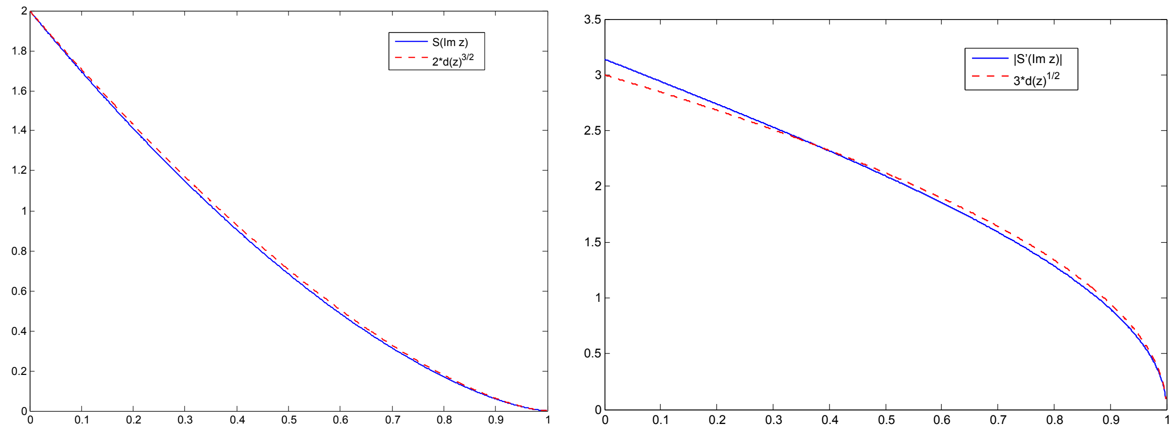

Figure 1.4: To illustrate Proposition 1.2.3 we show on the left hand side $S(\operatorname{Im} z)$, for $g(x) = e^{-ix}$, compared to $2 \cdot d(z)^{3/2}$ for $0 \le \operatorname{Im} z \le 1$. Due to the above choice of g we have that here $d(z) = (1 - \operatorname{Im} z)$ for $z \in \Sigma \cap \{z \in \mathbb{C};\ \operatorname{Im} z \ge 0\}$ (cf. (1.1.13)). Similarly, we show on the right hand side $|\partial_{\operatorname{Im} z} S(\operatorname{Im} z)|$ compared to $3 \cdot d(z)^{1/2}$.

Remark 1.2.4. Note that in (1.2.2) we chose to define $\partial_{\operatorname{Im} z} S(z) := x_-(z) - x_+(z)$ for $\operatorname{Im} z = \langle \operatorname{Im} g \rangle$. We will keep this definition throughout this text. Furthermore, we will keep the definition $d(z) := \operatorname{dist}(z, \partial\Sigma)$ throughout this entire work.

With the convention $\|(P_h - z)^{-1}\| = \infty$ for $z \in \sigma(P_h)$ we have the following estimate on the resolvent growth of P_h:

Proposition 1.2.5. *Let $g(x)$ be as above. For $z \in \mathbb{C}$ and $h > 0$ define,*

$$\Phi(z,h) := \begin{cases} -\frac{2\pi i}{h}(z - \langle g \rangle), & \text{if } \operatorname{Im} z < \langle \operatorname{Im} g \rangle, \\ \frac{2\pi i}{h}(z - \langle g \rangle), & \text{if } \operatorname{Im} z > \langle \operatorname{Im} g \rangle, \end{cases}$$

where $\operatorname{Re}\Phi(z,h) \le 0$. Then, under the assumptions of Definition 1.2.2 we have for $z \in \Omega \Subset \Sigma$ as in (1.2.1) that

$$\begin{aligned} \|(P_h - z)^{-1}\| &= \frac{\sqrt{\pi}\left|1 - e^{\Phi(z,h)}\right|^{-1} e^{\frac{S(z)}{h}}}{\sqrt{h}\left(\frac{i}{2}\{p,\overline{p}\}(\rho_+)\frac{i}{2}\{\overline{p},p\}(\rho_-)\right)^{\frac{1}{4}}}(1 + \mathcal{O}(h)) \qquad (1.2.3)\\ &\asymp \frac{e^{\frac{S(z)}{h}}}{\sqrt{h}\, d(z)^{1/4}}, \ \textit{for } |\operatorname{Im} z - \langle \operatorname{Im} g \rangle| > 1/C,\ C \gg 1, \end{aligned}$$

where $\left|1 - e^{\Phi(z,h)}\right| = 0$ if and only if $z \in \sigma(P_h)$. Moreover,

$$\left|1 - e^{\Phi(z,h)}\right| = 1 + \mathcal{O}\left(e^{-\frac{2\pi}{h}|\operatorname{Im} z - \langle \operatorname{Im} g \rangle|}\right).$$

This proposition will be proven in Section 2.8.1. The growth of the norm of the resolvent away from the line $\operatorname{Im} z = \langle \operatorname{Im} g \rangle$ is exponential and determined by the function $S(z)$. A similar result valid for $z \in \Gamma_\tau$ with $h^{2/3} \ll \tau \ll 1$ (cf (1.1.18)) has been obtained by W. Bordeaux-Montrieux [4, 5].

It will be very useful to write the coupling constant δ as follows:

Hypothesis 1.2.6. For $h > 0$, define

$$\delta := \delta(h) := \sqrt{h} e^{-\frac{\epsilon_0(h)}{h}}$$

with $\left(\kappa - \frac{1}{2}\right) h \ln(h^{-1}) + Ch \le \epsilon_0(h) < S(\langle \operatorname{Im} g \rangle)$ for some $\kappa > 0$ and $C > 0$ large and where the last inequality is uniform in $h > 0$. This is equivalent to the bounds

$$\sqrt{h} e^{-\frac{S(\langle \operatorname{Im} g \rangle)}{h}} < \delta \ll h^\kappa.$$

Remark 1.2.7. The upper bound on $\epsilon_0(h)$ has been chosen in order to produce eigenvalues sufficiently far away from the line $\operatorname{Im} z = \langle \operatorname{Im} g \rangle$ where we find $\sigma(P_h)$. The lower bound on $\epsilon_0(h)$ is needed because we want to consider small random perturbations with respect to P_h (cf. (1.1.12) and (1.1.16)).

1.2.2 – Auxiliary operator.

To describe the elements of the average density of eigenvalues, it will be very useful to introduce the following operators which have already been used in the study of the spectrum of P_h^δ by Sjöstrand [67]. For the readers convenience, we will give a short overview:

Let $z \in \mathbb{C}$ and we define the following z-dependent elliptic self-adjoint operators

$$\begin{aligned} &Q(z), \tilde{Q}(z): \ L^2(S^1) \to L^2(S^1) \quad \text{where} \\ &Q(z) := (P_h - z)^*(P_h - z), \quad \tilde{Q}(z) := (P_h - z)(P_h - z)^* \end{aligned} \tag{1.2.4}$$

with domains $\mathscr{D}(Q(z)), \mathscr{D}(\tilde{Q}(z)) = H^2(S^1)$. Since S^1 is compact and these are elliptic, non-negative, self-adjoint operators their spectra are discrete and contained in the interval $[0, \infty[$. Since

$$Q(z)u = 0 \Rightarrow (P_h - z)u = 0$$

it follows that $\mathscr{N}(Q(z)) = \mathscr{N}(P_h - z)$ and $\mathscr{N}(\tilde{Q}(z)) = \mathscr{N}((P_h - z)^*)$. Furthermore, if $\lambda \neq 0$ is an eigenvalue of $Q(z)$ with corresponding eigenvector e_λ we see that $f_\lambda := (P_h - z)e_\lambda$ is an eigenvector of $\tilde{Q}(z)$ with the eigenvalue λ. Similarly, every non-vanishing eigenvalue of $\tilde{Q}(z)$ is an eigenvalue of $Q(z)$ and moreover, since $P_h - z$, $(P_h - z)^*$ are Fredholm operators of index 0 we see that

$$\dim \mathscr{N}(P_h - z) = \dim \mathscr{N}((P_h - z)^*).$$

Hence the spectra of $Q(z)$ and $\tilde{Q}(z)$ are equal

$$\sigma(Q(z)) = \sigma(\tilde{Q}(z)) = \{t_0^2, t_1^2, \dots\}, \ 0 \leq t_j \nearrow \infty. \tag{1.2.5}$$

We will show in Proposition 2.1.7 that for $z \in \Omega \Subset \Sigma$ (cf (1.2.1))

$$t_0^2(z) \leq \mathscr{O}\left(d(z)^{\frac{1}{2}} h e^{-\frac{2S}{h}}\right), \quad t_1^2(z) \geq \frac{d(z)^{\frac{1}{2}} h}{\mathscr{O}(1)}. \tag{1.2.6}$$

Now consider the orthonormal basis of $L^2(S^1)$

$$\{e_0, e_1, \dots\} \tag{1.2.7}$$

consisting of the eigenfunctions of $Q(z)$. By the previous observations we have

$$(P_h - z)(P_h - z)^*(P_h - z)e_j = t_j^2 (P_h - z)e_j.$$

Thus defining f_0 to be the normalized eigenvector of $\tilde{Q}$ corresponding to the eigenvalue t_0^2 and the vectors $f_j \in L^2(S^1)$, for $j \in \mathbb{N}$, as the normalization of $(P_h - z)e_j$ such that

$$(P_h - z)e_j = \alpha_j f_j, \quad (P_h - z)^* f_j = \beta_j e_j \quad \text{with } \alpha_j \beta_j = t_j^2, \tag{1.2.8}$$

yields an orthonormal basis of $L^2(S^1)$

$$\{f_0, f_1, \dots\} \tag{1.2.9}$$

consisting of the eigenfunctions of $\tilde{Q}(z)$. Since

$$\alpha_j = ((P_h - z)e_j | f_j) = (e_j | (P_h - z)^* f_j) = \overline{\beta}_j$$

we can conclude that $\alpha_j \overline{\alpha}_j = t_j^2$.

It is clear from (1.2.6), (1.2.8) that $e_0(z)$ (resp. $f_0(z)$) is an exponentially accurate quasimode for $P_h - z$ (resp. $(P_h - z)^*$). We will see in Section 2.1 that it is localized to $\rho_+(z)$ (resp. $\rho_-(z)$). We will prove in the Sections 2.2.2 and 2.2.4 the following two formulas for the tunneling effect:

Proposition 1.2.8. *Let $z \in \Omega \Subset \Sigma$ be as in* (1.2.1) *and let e_0 and f_0 be as in (1.2.7) and in (1.2.9). Furthermore, let S be as in Definition 1.2.2, let p be as in* (1.1.7) *and $\rho_\pm$ be as in* (1.1.14). *Let $h^{\frac{2}{3}} \ll d(z)$, then for all $z \in \Omega$ with $|\mathrm{Im}\, z - \langle \mathrm{Im}\, g\rangle| > 1/C$, $C \gg 1$,*

$$|(e_0|f_0)| = \frac{\left(\frac{i}{2}\{p,\overline{p}\}(\rho_+)\frac{i}{2}\{\overline{p},p\}(\rho_-)\right)^{\frac{1}{4}}}{\sqrt{\pi h}} |\partial_{\mathrm{Im}\, z} S(z)|\,(1+K(z;h))\,\mathrm{e}^{-\frac{S(z)}{h}},$$

where $K(z;h)$ depends smoothly on z and satisfies for all $\beta \in \mathbb{N}^2$ that

$$\partial^{\beta}_{z\overline{z}} K(z;h) = \mathcal{O}\left(d(z)^{\frac{|\beta|}{2}-\frac{3}{4}} h^{-|\beta|+\frac{1}{2}}\right).$$

Proposition 1.2.9. *Under the same assumptions as in Proposition 1.2.8, let $\chi \in \mathscr{C}_0^\infty(S^1)$ with $\chi \equiv 1$ in a small open neighborhood of $\overline{\{x_-(z):\ z \in \Omega\}}$. Then, for $h^{\frac{2}{3}} \ll d(z)$,*

$$|([P_h,\chi]e_0|f_0)| = \sqrt{h}\left(\frac{\frac{i}{2}\{p,\overline{p}\}(\rho_+)\frac{i}{2}\{\overline{p},p\}(\rho_-)}{\pi^2}\right)^{\frac{1}{4}} (1+K(z;h))\,\mathrm{e}^{-\frac{S(z)}{h}},$$

where $K(z;h)$ depends smoothly on z and satisfies for all $\beta \in \mathbb{N}^2$ that

$$\partial^{\beta}_{z\overline{z}} K(z;h) = \mathcal{O}\left(d(z)^{\frac{|\beta|-3}{2}} h^{1-(|\beta|)}\right).$$

1.2.3 – Average density of eigenvalues.

We begin by defining the point process of eigenvalues of the perturbed operator P_h^δ (cf Hypothesis 1.1.3).

Definition 1.2.10. Let P_h^δ be as in Hypothesis 1.1.3, then we define the point process

$$\Xi := \sum_{z \in \sigma(P_h^\delta)} \delta_z, \tag{1.2.10}$$

where the eigenvalues are counted according to their multiplicities and δ_z denotes the Dirac-measure at z.

Ξ is a well-defined random measure (cf. for example [11]) since, for $h > 0$ small enough, P_h^δ is a random operator with discrete spectrum. To obtain an h-asymptotic formula for the average density of eigenvalues, we are interested in intensity measure of Ξ (with respect to the restriction in the random variables, see Hypothesis 1.1.6), i.e. the measure μ_1 defined by

$$T_1(\varphi) := \mathbb{E}\left[\Xi(\varphi)\mathbb{1}_{B(0,R)}\right] = \int_{\mathbb{C}} \varphi(z)\, d\mu_1(z)$$

for all $\varphi \in \mathscr{C}_0(\Omega)$ with $\Omega \Subset \Sigma$ as in Hypothesis 1.1.7. The measure μ_1 is well defined since T_1 is a positive linear functional on $\mathscr{C}_0(\Omega)$.

Remark 1.2.11. Such an approach is employed with great success in the study of zeros of random polynomials and Gaussian analytic functions; we refer the reader to the works of B. Shiffman and S. Zelditch [61, 62, 60, 59], M. Sodin [75] an the book [38] by J. Hough, M. Krishnapur, Y. Peres and B. Virág.

Our main result giving the average density of eigenvalues of P_h^δ is the following:

Theorem 1.2.12. *Let $\Omega \Subset \Sigma$ be as in Hypothesis 1.1.7. Let $C > 0$ be as in* (1.1.11) *and let $C_1 > 0$ be as in (1.1.10) such that $C - C_1 > 0$ is large enough. Let $\delta > 0$ be as in Hypothesis 1.2.6 with $\kappa > 4$ large enough. Define $N := (2\lfloor C_1/h\rfloor + 1)^2$ and let $B(0,R) \subset \mathbb{C}^N$ be the ball of radius $R := Ch^{-1}$ centered at zero. Then, there exists a $C_2 > 0$ such that for $h > 0$ small enough and for all $\varphi \in \mathscr{C}_0(\Omega)$*

$$\mathbb{E}\left[\ \Xi(\varphi)\mathbb{1}_{B(0,R)}\ \right] = \int \varphi(z) D(z,h,\delta) L(dz) + \mathcal{O}\left(\mathrm{e}^{-\frac{C_2}{h^2}}\right), \tag{1.2.11}$$

with the density

$$D(z,h,\delta) = \frac{1+\mathcal{O}\Big(\delta h^{-\frac{3}{2}} d(z)^{-1/4}\Big)}{\pi}\Psi(z;h,\delta)\exp\{-\Theta(z;h,\delta)\}, \tag{1.2.12}$$

which depends smoothly on z and is independent of φ. Moreover, $\Psi(z;h,\delta) = \Psi_1(z;h) + \Psi_2(z;h,\delta)$ and for $z \in \Omega$ with $d(z) \gg (h\ln h^{-1})^{2/3}$

$$\begin{aligned}
\Psi_1(z;h) &= \frac{1}{h}\left\{\frac{i}{\{p,\overline{p}\}(\rho_+(z))} + \frac{i}{\{\overline{p},p\}(\rho_-(z))}\right\} + \mathcal{O}\big(d(z)^{-2}\big),\\
\Psi_2(z;h,\delta) &= \frac{\big|(e_0|f_0)\big|^2}{\delta^2}\big(1+\mathcal{O}\big(d(z)^{-3/4}h^{1/2}\big)\big),\\
\Theta(z;h,\delta) &= \frac{\big|([P_h,\chi]e_0|f_0) + \mathcal{O}\big(d(z)^{-1/4}h^{-5/2}\delta^2\big)\big|^2}{\delta^2(1+\mathcal{O}(h^\infty))}\left(1+\mathcal{O}\left(\mathrm{e}^{-\frac{\asymp d(z)^{3/2}}{h}}\right)\right).
\end{aligned} \tag{1.2.13}$$

Furthermore, in (1.2.11), $\mathcal{O}\left(\mathrm{e}^{-\frac{C_2}{h^2}}\right)$ means $\langle T_h,\varphi\rangle$, where $T_h \in \mathcal{D}'(\mathbb{C})$ such that

$$|\langle T_h,\varphi\rangle| \leq C\|\varphi\|_\infty \mathrm{e}^{-\frac{C_2}{h^2}}$$

for all $\varphi \in \mathscr{C}_0(\Omega)$ where $C > 0$ is independent of h, δ and φ.

Let us give some comments on this result. The dominant part of the density of eigenvalues D consists of three parts: the first, Ψ_1, is up to a small error the Lebesgue density of $p_*(d\xi\wedge dx)$, where $d\xi\wedge dx$ is the symplectic form on T^*S^1 and p is as in (1.1.7). We prove in Proposition 2.4.2 that

$$p_*(d\xi\wedge dx) = \sigma(z)L(dz), \quad \text{with } \sigma(z) := \left(\frac{2i}{\{p,\overline{p}\}(\rho_+(z))} + \frac{2i}{\{\overline{p},p\}(\rho_-(z))}\right). \tag{1.2.14}$$

The second part, Ψ_2, is given by a tunneling effect. Inside the $(\sqrt{h}\delta)$-pseudospectrum its contribution can be absorbed in the error term of Ψ_1. However, close to the boundary of the δ-pseudospectrum Ψ_2 becomes of order h^{-2} and thus yields a higher density of eigenvalues. This can be seen by comparing the more explicit formula for Ψ_2 given in Proposition 1.2.13 with the expression for the norm of the resolvent of P_h given in Proposition 1.2.5. More details on the form of Ψ_2 in this zone will be given in Proposition 1.2.17.

The third part, $\exp\{-\Theta\}$, is also given by a tunneling effect and it plays the role of a cut-off function which exhibits double exponential decay outside the δ-pseudospectrum and is close to 1 inside. This will be made more precise in Section 1.2.4.

We have the following explicit formulas for these functions and their growth properties:

Proposition 1.2.13. *Under the assumptions of Definition 1.2.2 and Theorem 1.2.12, define for $h > 0$ and $\delta > 0$ the functions*

$$\Theta^0(z;h,\delta) := \frac{h\left(\frac{i}{2}\{p,\overline{p}\}(\rho_+)\frac{i}{2}\{\overline{p},p\}(\rho_-)\right)^{\frac{1}{2}}}{\pi}\frac{\mathrm{e}^{-\frac{2S}{h}}}{\delta^2}.$$

Then, for $|\mathrm{Im}\, z - \langle \mathrm{Im}\, g\rangle| > 1/C$, $C \gg 1$,

$$\begin{aligned}
\Psi_2(z;h,\delta) &= \frac{\left(\frac{i}{2}\{p,\overline{p}\}(\rho_+)\frac{i}{2}\{\overline{p},p\}(\rho_-)\right)^{\frac{1}{2}}}{\pi h\delta^2\exp\{\frac{2S}{h}\}}|\partial_{\mathrm{Im}\, z}S(z)|^2\left(1+\mathcal{O}\left(\frac{h^{1/2}}{d(z)^{\frac{3}{4}}}\right)\right)\\
\Theta(z;h,\delta) &= \Theta^0(z;h,\delta)\left(1+\mathcal{O}\left(\frac{h^{\frac{3}{2}}}{d(z)^{\frac{1}{4}}}\right)\right) + \mathcal{O}\left(\frac{d(z)^{\frac{1}{4}}\delta}{h^2} + \frac{\delta^2}{d(z)^{\frac{1}{2}}h^5}\right).
\end{aligned} \tag{1.2.15}$$

The estimates in (1.2.15) are stable under application of $d(z)^{-\frac{|\beta|}{2}}h^{|\beta|}\partial^\beta_{z\overline{z}}$, for $\beta \in \mathbb{N}^2$.

Proposition 1.2.14. *Under the assumptions of Definition 1.2.2 and Theorem 1.2.12 we have that*

$$\frac{i}{2}\{p,\overline{p}\}(\rho_+)\frac{i}{2}\{\overline{p},p\}(\rho_-) \asymp d(z), \quad \frac{i}{\{p,\overline{p}\}(\rho_+(z))} + \frac{i}{\{\overline{p},p\}(\rho_-(z))} \asymp \frac{1}{\sqrt{d(z)}}$$

and

$$\Psi_2(z;h,\delta) \asymp \frac{(d(z))^{3/2}\mathrm{e}^{-\frac{2S}{h}}}{h\delta^2}, \quad \Theta^0(z;h,\delta) \asymp h\sqrt{d(z)}\left|1-\mathrm{e}^{\Phi(z,h)}\right|\frac{\mathrm{e}^{-\frac{2S}{h}}}{\delta^2}.$$

In the next Subsection we will explain the asymptotic properties of the density appearing in (1.2.11).

1.2.4 – Properties of the average density of eigenvalues and its integral with respect to $\operatorname{Im} z$

It will be sufficient for our purposes to consider rectangular subsets of Σ: for $c < d$ define

$$\Sigma_{c,d} := \left\{z \in \Sigma \;\middle|\; \min_{x\in S^1} \operatorname{Im} g(x) \leq \operatorname{Im} z \leq \max_{x\in S^1} \operatorname{Im} g(x),\ c < \operatorname{Re} z < d\right\}. \tag{1.2.16}$$

Roughly speaking, there exist three regions in Σ:

$$\begin{aligned}
&(1) \quad z \in \Sigma^W \subset \Sigma \iff \|(P_h - z)^{-1}\| \gg (\sqrt{h}\delta)^{-1},\\
&(2) \quad z \in \Sigma^R \subset \Sigma \iff \|(P_h - z)^{-1}\| \asymp \delta^{-1},\\
&(3) \quad z \in \Sigma^V \subset \Sigma \iff \|(P_h - z)^{-1}\| \ll \delta^{-1},
\end{aligned}$$

which depend on the strength of the coupling constant $\delta > 0$. In Σ^W, the average density is of order h^{-1} and is governed by the symplectic volume yielding a Weyl law. In Σ^R, the average density spikes and Ψ_2 becomes the leading term and is of order h^{-2} and it yields in total a Poisson-type distribution, cf. Proposition 1.2.17. In Σ^V, the average density is rapidly decaying, since

$$\Theta \asymp \|(P_h - z)^{-1}\|^{-2}\delta^{-2},$$

which follows from Proposition 1.2.9 and Proposition 1.2.5.

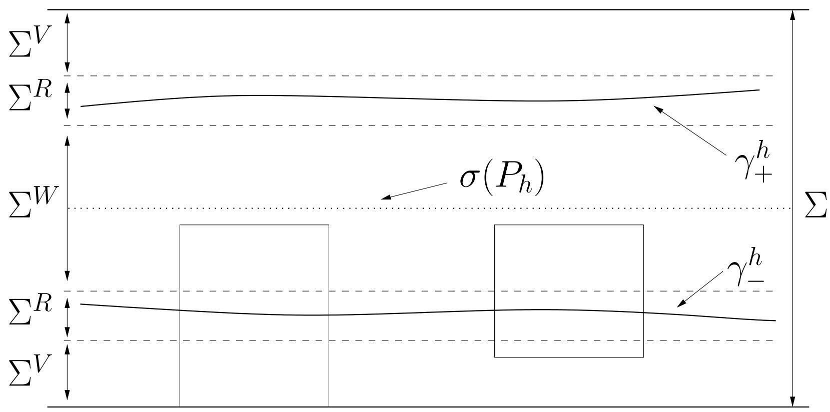

Figure 1.5: The three zones in Σ with a schematic representation of $\gamma_\pm^h$. The two boxes indicate zones where the integrated densities are equal up to a small error.

We will prove that there exist two smooth curves, $\Gamma_\pm^h$, close to the boundary of the δ-pseudo-spectrum of P_h^δ, along which the average density of eigenvalues obtains its local maxima. Note that this is still inside the $(Ch^{-1}\delta)$-pseudospectrum of P_h^δ (cf Hypothesis 1.1.6) since pseudospectra are nested (meaning that $\sigma_{\varepsilon_1}(P_h^\delta) \subset \sigma_{\varepsilon_2}(P_h^\delta)$ for $\varepsilon_1 < \varepsilon_2$).

Proposition 1.2.15. *Let $z \in \Omega \Subset \Sigma_{c,d}$ with $\Sigma_{c,d}$ as in (1.2.16), let $S(z)$ be as in Definition 1.2.2 and let $t_0^2(z)$ be as in (1.2.5). Let $\delta > 0$ and $\varepsilon_0(h)$ be as in Hypothesis 1.2.6 with $\kappa > 4$ large enough. Moreover, let $D(z,h,\delta)$ be the average density of eigenvalues of the operator of P_h^δ given in Theorem 1.2.12. Then,*

1. *for $0 < h \ll 1$, there exist numbers $y_\pm(h)$ such that $\varepsilon_0(h) = S(y_\pm(h))$ with*
$$\frac{1}{C}\left(h\ln h^{-1}\right)^{\frac{2}{3}} \ll y_-(h) < \langle \operatorname{Im} g\rangle - ch\ln h^{-1}$$
$$< \langle \operatorname{Im} g\rangle + ch\ln h^{-1} < y_+(h) \ll \operatorname{Im} g(b) - \frac{1}{C}\left(h\ln h^{-1}\right)^{\frac{2}{3}},$$
for some constants $C, c > 1$. Furthermore,
$$y_-(h), (\operatorname{Im} g(b) - y_+(h)) \asymp (\varepsilon_0(h))^{2/3};$$

2. *there exists $h_0 > 0$ and a family of smooth curves, indexed by $h \in]h_0, 0[$,*
$$\gamma_\pm^h :]c,d[\longrightarrow \mathbb{C} \text{ with } \operatorname{Re}\gamma_\pm^h(t) = t$$
such that
$$|t_0(\gamma_\pm^h(t))| = \delta.$$
Moreover,
$$\|(P_h - \gamma_\pm^h(t))^{-1}\| = \delta^{-1},$$
and
$$\operatorname{Im}\gamma_\pm^h(\operatorname{Re} z) = y_\pm(\varepsilon_0(h))\left(1 + \mathcal{O}\left(\frac{h}{\varepsilon_0(h)}\right)\right).$$
Furthermore, there exists a constant $C > 0$ such that
$$\frac{d\operatorname{Im}\gamma_\pm^h}{dt}(t) = \mathcal{O}\left(\exp\left[-\frac{\varepsilon_0(h)}{Ch}\right]\right).$$

3. *there exists $h_0 > 0$ and a family of smooth curves, indexed by $h \in]h_0, 0[$,*
$$\Gamma_\pm^h :]c,d[\longrightarrow \mathbb{C},\ \operatorname{Re}\Gamma_\pm^h(t) = t,$$
with $\Gamma_- \subset \{\operatorname{Im} z < \langle \operatorname{Im} g\rangle\}$ and $\Gamma_+ \subset \{\operatorname{Im} z > \langle \operatorname{Im} g\rangle\}$, along which $\operatorname{Im} z \mapsto D(z,h)$ takes its local maxima on the vertical line $\operatorname{Re} z = \text{const.}$ and
$$\frac{d}{dt}\operatorname{Im}\Gamma_\pm^h(t) = \mathcal{O}\left(\frac{h^4}{\varepsilon_0(h)^4}\right).$$
Moreover, for all $c < t < d$
$$|\Gamma_\pm^h(t) - \gamma_\pm^h(t)| \leq \mathcal{O}\left(\frac{h^5}{\varepsilon_0(h)^{13/3}}\right).$$

With respect to the above described curves we prove the following properties of the average density of eigenvalues:

Proposition 1.2.16. *Let $d\xi \wedge dx$ be the symplectic form on T^*S^1 and p as in (1.1.7). Let $\varepsilon_0 = \varepsilon_0(h)$ be as in Hypothesis 1.2.6. Then, under the assumptions of Theorem 1.2.12 there exist $\alpha, \beta > 0$ such that*

1. *for* $z \in \Sigma_{c,d}$ *with*

$$\operatorname{Im}\gamma_-(\operatorname{Re} z) + \alpha \frac{h}{\varepsilon_0^{1/3}} \ln \frac{\varepsilon_0^{1/3}}{h} \leq \operatorname{Im} z \leq \operatorname{Im}\gamma_+(\operatorname{Re} z) - \alpha \frac{h}{\varepsilon_0^{1/3}} \ln \frac{\varepsilon_0^{1/3}}{h}$$

we have that

$$D(z;h,\delta)L(dz) = \frac{1}{2\pi h} p_*(d\xi \wedge dx) + \mathcal{O}\left(d(z)^{-2}\right) L(dz),$$

where $D(z;h,\delta)$ *is the average density of eigenvalues of the operator of* P_h^δ *given in Theorem 1.2.12.*

2. *for*

$$\Omega_1(\beta) := \left\{ z \in \Sigma_{c,d} \,\middle|\, \operatorname{Im}\gamma_-(\operatorname{Re} z) - \frac{h}{\varepsilon_0^{1/3}} \ln\left(\beta \ln \frac{\varepsilon_0^{1/3}}{h}\right) \right.$$
$$\left. \leq \operatorname{Im} z \leq \operatorname{Im}\gamma_+(\operatorname{Re} z) + \frac{h}{\varepsilon_0^{1/3}} \ln\left(\beta \ln \frac{\varepsilon_0^{1/3}}{h}\right) \right\},$$

we have that

$$\int\limits_{z \in \Omega_1(\beta)} D(z;h,\delta)L(dz) = \int\limits_{\Sigma_{c,d}} \frac{p_*(d\xi \wedge dx)}{2\pi h} + \mathcal{O}\left(\varepsilon_0^{-\frac{2}{3}}\right).$$

3. *for all* $\varepsilon > 0$ *and all* $\Omega(\varepsilon) \subset \Sigma_{c,d} \backslash \Omega_2(\beta,\varepsilon)$ *satisfying Hypothesis 1.1.7, where*

$$\Omega_2(\beta,\varepsilon) := \left\{ z \in \Sigma_{c,d} \,\middle|\, \operatorname{Im}\gamma_-(\operatorname{Re} z) - \frac{h}{\varepsilon_0^{1/3}} \ln\left(\beta \ln \frac{\varepsilon_0^{1/3}}{h}\right) - \varepsilon \right.$$
$$\left. \leq \operatorname{Im} z \leq \operatorname{Im}\gamma_+(\operatorname{Re} z) + \frac{h}{\varepsilon_0^{1/3}} \ln\left(\beta \ln \frac{\varepsilon_0^{1/3}}{h}\right) + \varepsilon \right\},$$

we have that

$$\int_{\Omega(\varepsilon)} D(z;h,\delta)L(dz) = \mathcal{O}\left(\exp\left\{-\mathrm{e}^{\frac{\varepsilon}{Ch}}\right\}\right).$$

Proposition 1.2.16 makes more precise the rough description of the behavior of the average density of eigenvalues, given at the beginning of this section: Point 1. tells us that in the interior of the δ-pseudospectrum, up to a distance of order $h \ln \frac{1}{h}$ to the curves $\gamma_\pm^h$ (see Figure 1.5), the density is given by a Weyl law. Assertion 2. tells us that the eigenvalues accumulate strongly in the close vicinity of these curves such that when integrating the density in the box $\Omega_1 \Subset \Sigma_{c,d}$ the number of eigenvalues is given (up to small error) by the integrated Weyl density in all of $\Sigma_{c,d}$ (cf Figure 1.5). This augmented density can be seen as the accumulated eigenvalues which would have been given by a Weyl law in the region from $\gamma_\pm^h$ up to the boundary $\partial\Sigma$ (see also Figures 1.6 and 1.7 for an example).

The last point of the proposition tells us that outside of a strip of the form of Ω_1 the density decays double-exponentially.

The density in the zone of spectral accumulation We give a finer description of the density of eigenvalues close to its local maxima at $\Gamma_\pm^h$:

Proposition 1.2.17. *Assume the hypotheses of Theorem 1.2.12. Let* $S(z)$ *be as in Definition 1.2.2 and let* $\Psi_2(z;h,\delta)$ *and* $\Theta(z;h,\delta)$ *be as in Theorem 1.2.12. Then for* $|\operatorname{Im} z - \langle \operatorname{Im} g \rangle| > 1/C$ *with* $C \gg 1$ *large enough,*

$$\Psi_2(z;h,\delta)\mathrm{e}^{-\Theta(z;h,\delta)} = \left[\frac{|\partial_{\operatorname{Im} z} S(z)|^2}{h^2} \Theta(z;h,\delta)\left(1 + \mathcal{O}\left(d(z)^{-3/4} h^{1/2}\right)\right) + \mathcal{O}\left(d(z)^{5/4}\right)\right] \mathrm{e}^{-\Theta(z;h,\delta)}.$$

Let us give some remarks on this result. First, we see that we can approximate the second part of the density of eigenvalues by a Poisson type distribution. Second, since $\Theta \asymp \|(P_h - z)^{-2}\|^{-1}\delta^{-2}$, we see that the effects of the second part of the density vanish in the error term of Ψ_1 as long as $\|(P_h - z)^{-1}\| \gg (\sqrt{h}\delta)^{-1}$. However, for $\|(P_h - z)^{-1}\| \asymp \delta^{-1}$ it is of order $\mathcal{O}(d(z)h^{-2})$ and dominates the Weyl term.

1.2.5 – Example: Numerical simulations

To illustrate our results we look at the discretization of $P_h = hD + \mathrm{e}^{-ix}$ in Fourier space which is approximated by the $(2N+1) \times (2N+1)$-matrix $H = hD + E$, $N \in \mathbb{N}$, where D and E are defined by

$$D_{j,k} := \begin{cases} j & \text{if } j = k, \\ 0 & \text{else} \end{cases} \quad \text{and} \quad E_{j,k} := \begin{cases} 1 & \text{if } k = j+1, \\ 0 & \text{else,} \end{cases}$$

where $j, k \in \{-N, -N+1, \dots, N\}$. Let R be a $(2N+1) \times (2N+1)$ random matrix, where the entries $R_{j,k}$ are independent and identically distributed complex Gaussian random variables, $R_{j,k} \sim \mathcal{N}_{\mathbb{C}}(0,1)$. For $h > 0$ and $\delta > 0$ as in Theorem 1.2.12, we let MATLAB calculate the spectrum $\sigma(H + \delta R)$. Since here $g(x) = \mathrm{e}^{-ix}$ (cf. (1.1.6)), it follows that in this case Σ is given by $\{z \in \mathbb{C};\ |\mathrm{Im}\, z| \le 1\}$ (cf. (1.1.13)).

Remark 1.2.18. Details regarding the MATLAB code used to obtain these simulations can be found in Appendix A.

We are going to perform our numerical experiments for the following two cases:

Polynomially small (in h) coupling δ We set the above parameters to be $h = 2 \cdot 10^{-3}$, $\delta = 2 \cdot 10^{-12} \approx 0.1 \cdot h^4$ and $N = 1999$. Figure 1.6 shows the spectrum of $H + \delta R$ computed by MATLAB.

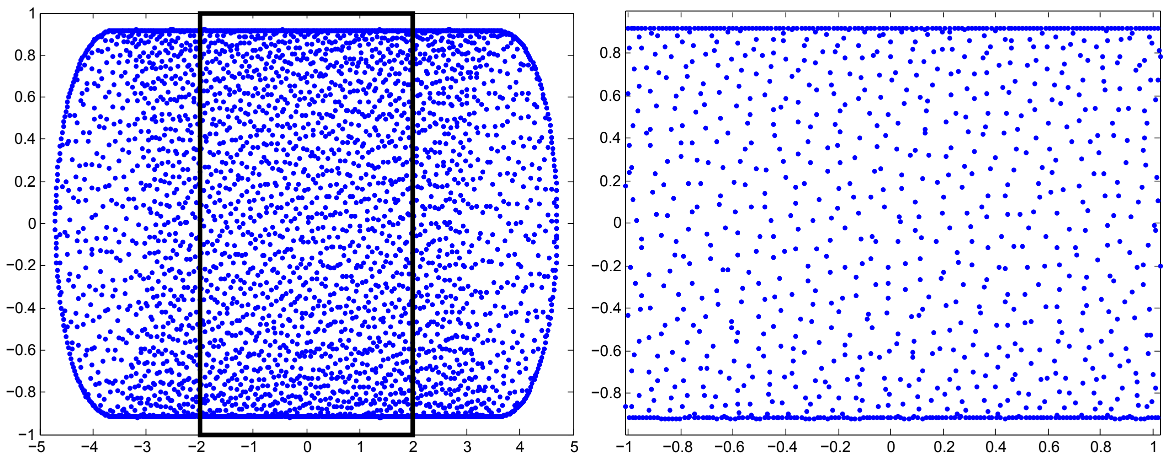

Figure 1.6: On the left hand side we present the spectrum of the discretization of $hD + \exp(-ix)$ (approximated by a 3999×3999-matrix) perturbed with a random Gaussian matrix δR with $h = 2 \cdot 10^{-3}$ and $\delta = 2 \cdot 10^{-12}$. The black box indicates the region where we count the number of eigenvalues to obtain Figure1.7. The right hand side is a magnification of the central part of the spectrum depicted on the left hand side.

The black box indicates the region where we count the number of eigenvalues to obtain the density of eigenvalues presented in Figure 1.7. Outside this box the influence from the boundary effects from our N-dimensional matrix are too strong. Figure 1.7 compares the experimental (given by counting the number of eigenvalues in the black box restricted to $\mathrm{Im}\, z \ge 0$ and averaging over 400 realizations of random Gaussian matrices) and the theoretical (cf Theorem 1.2.12) density and integrated density of eigenvalues.

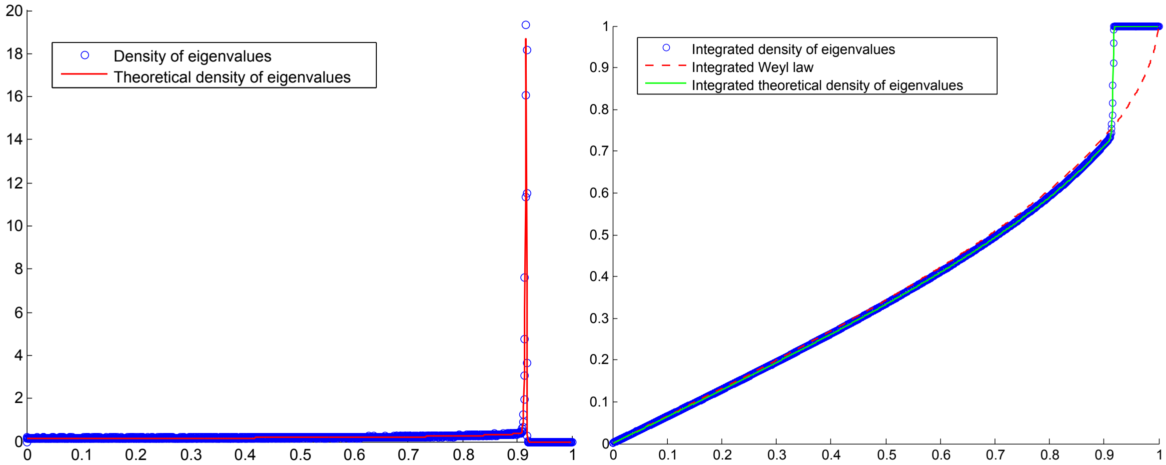

Figure 1.7: On the left hand side we compare the experimental and the theoretical (cf. Theorem 1.2.12) density of eigenvalues. On the right hand side we compare the experimental and the theoretical integrated density of eigenvalues with the integrated Weyl law. Here $h = 2 \cdot 10^{-3}$ and $\delta = 2 \cdot 10^{-12}$.

Exponentially small (in h) coupling δ We set the above parameters to be $h = 5 \cdot 10^{-2}$, $\delta = \exp(-1/h)$ and $N = 1000$. Figure 1.8 shows the spectrum of $H + \delta R$ computed by MATLAB. Similar to the

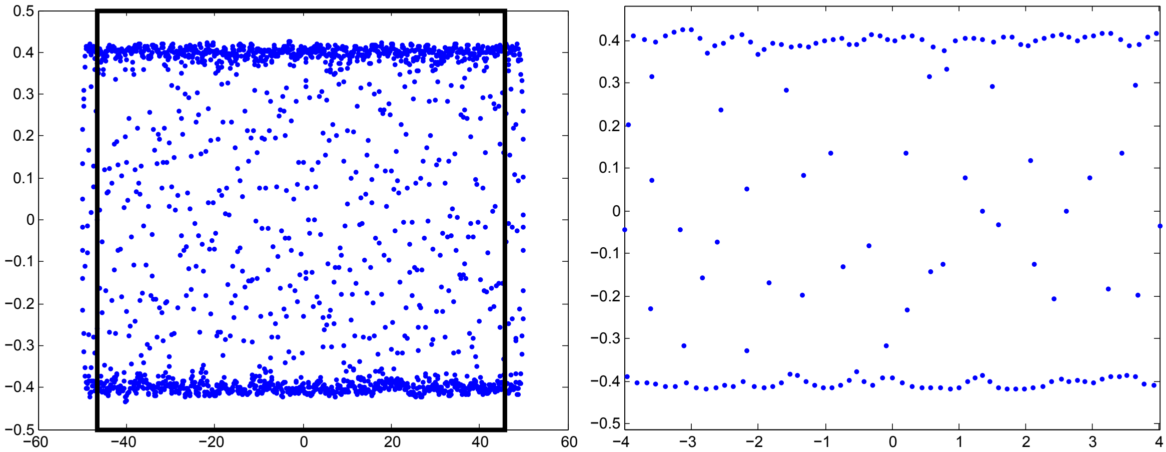

Figure 1.8: On the left hand side we present the spectrum of the discretization of $hD + \exp(-ix)$ (approximated by a 1999×1999-matrix) perturbed with a random Gaussian matrix δR with $h = 5 \cdot 10^{-2}$ and $\delta = \exp(-1/h)$. The black box indicates the region where we count the number of eigenvalues to obtain Figure 1.9. The right hand side is a magnification of the central part of the spectrum depicted on the left hand side.

above, the black box indicates the region where we count the number of eigenvalues to obtain the density of eigenvalues presented in Figure 1.9. This figure compares the experimental (given by counting the number of eigenvalues in the black box restricted to $\operatorname{Im} z \geq 0$ and averaging over 400 realizations of random Gaussian matrices) and the theoretical (cf Theorem 1.2.12) density and integrated density of eigenvalues.

The Figures 1.6, 1.7, 1.8 and 1.9 confirm the theoretical result presented in Theorem 1.2.12 since the green lines, representing the plotted average density of eigenvalues given by Theorem 1.2.12, match perfectly the experimentally obtained density of eigenvalues. Furthermore, these figures show the three zones described in Section 1.2.4 (see also Proposition 1.2.16):

The *first zone*, is in the middle of the spectrum (cf. Figures 1.6, 1.8) corresponding to the zone where $\|(P_h - z)^{-1}\| \gg (\sqrt{h}\delta)^{-1}$. There we see roughly an aequidistribution of points at distance $\sqrt{h}$. The right hand side of Figures 1.7 and 1.9 shows that the number of eigenvalues in this zone

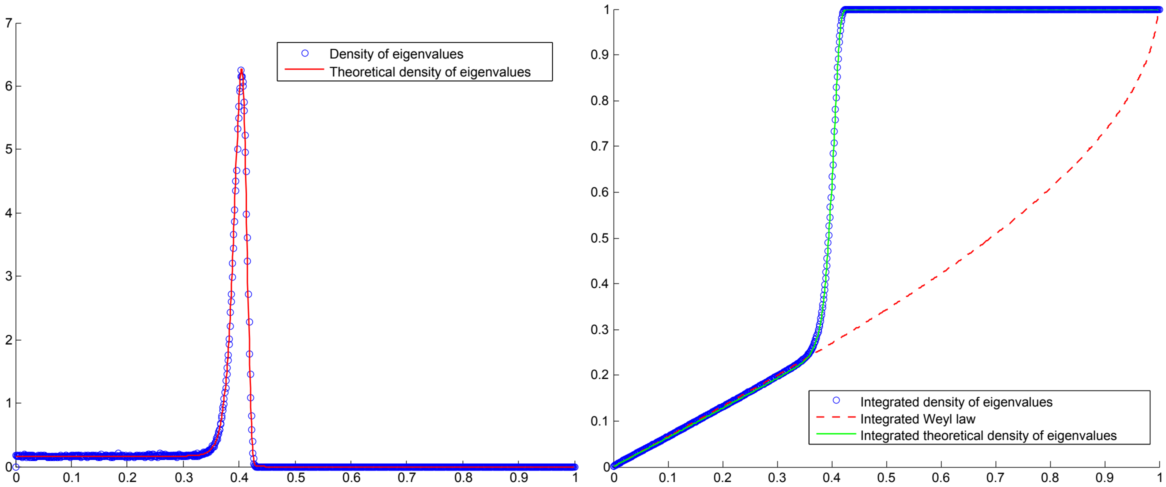

Figure 1.9: Experimental (each point represents the mean, over 1000 realizations, number of eigenvalues in a small box) vs predicted eigenvalue density (i.e. the principal terms of the average eigenvalue density given in Theorem 1.2.12) for $h = 5 \cdot 10^{-2}$ and $\delta = \exp(-1/h)$.

is given by a *Weyl law*, as predicted by Proposition 1.2.16.

When comparing Figure 1.7 and 1.9 we can see clearly that the Weyl law breaks down earlier when the coupling constant δ gets smaller. Indeed, when $\delta > 0$ is exponentially small in $h > 0$, the break down happens well in the interior of Σ, precisely as predicted by Proposition 1.2.16.

Another important property of this zone is that there is an increase in the density of the spectral points as we approach the boundary of Σ, see Figure 1.7. This is due to the fact that the density given by the Weyl law becomes more and more singular as we approach $\partial\Sigma$ (cf. Proposition 1.2.14).

We will find the *second zone* by moving closer to the "edge" of the spectrum, see Figure 1.6 and 1.8. It can be characterized as the zone where $\|(P_h - z)^{-1}\| \asymp \delta^{-1}$. Figures 1.7 and 1.9 show that there is a strong accumulation of the spectrum close to the boundary of the pseudospectrum. Furthermore, we see in the image on the right hand side of Figure 1.6 and of Figure 1.8 that the zone of accumulation of eigenvalues is in a small tube around roughly a straight line. This is exactly as predicted by Proposition 1.2.15 and Proposition 1.2.17. Finally, let us remark that when looking at the Figures 1.6 and 1.8, we note that in this zone the average distance between eigenvalues is much closer than in the first zone.

The *third zone* is between the spectral edge and the boundary of Σ where we find no spectrum at all. It can be characterized as the zone where $\|(P_h - z)^{-1}\| \ll \delta^{-1}$, a *void region* as described in Proposition 1.2.16 (cf. Figures 1.7 and 1.9).

Let us stress again that as δ gets smaller the zone of accumulation moves further into the interior of Σ, thus diminishing the zone determined by the Weyl law and increasing the zone void of eigenvalues. This effect is most drastic in the case of δ being exponentially small in h, see Figure 1.9.

1.3 | Two-point eigenvalue interaction of the eigenvalues in Hager's model

To study the two-point eigenvalue interaction we are interested in the second moment of the point process Ξ, see Definition 1.2.10. We begin by recalling some facts about second moments of point processes from [38, 11], using the example of Ξ. The second moment (with respect to the restriction of the random variables introduced in Hypothesis 1.1.6) of Ξ is defined by the positive linear functional on $\mathscr{C}_0(\Omega^2)$, T_2, defined by

$$T_2(\varphi) := \mathbb{E}\left[\sum_{z,w\in\sigma(P_h^\delta)} \varphi(z,w)\mathbb{1}_{B(0,R)}\right] = \int_{\mathbb{C}^2} \varphi(z,w)d\mu_2(z,w)$$

for all $\varphi \in \mathscr{C}_0(\Omega^2)$. Here, we choose $\Omega \Subset \mathring{\Sigma}$ to be a subset of the interior of Σ:

Hypothesis 1.3.1. We assume that there exists a $C > 1$ such that

$$\Omega \Subset \mathring{\Sigma} \text{ is open, convex, relatively compact and simply connected with } \operatorname{dist}(\Omega,\partial\Sigma) > \frac{1}{C}. \tag{1.3.1}$$

Furthermore, we assume

Hypothesis 1.3.2. The coupling constant $\delta > 0$ in (1.1.9) satisfies

$$\delta := \delta(h) := \sqrt{h}e^{-\frac{\epsilon_0(h)}{h}} \tag{1.3.2}$$

with $\left(\kappa - \frac{1}{2}\right) h\ln(h^{-1}) + Ch \le \epsilon_0(h) < \min_{z\in\overline{\Omega}} S(z)/C$ for some $\kappa > 7/2$ and $C > 0$ large and where the last inequality is uniform in $h > 0$. Equivalently, δ satisfies the inequality

$$\sqrt{h}\exp\left\{-\frac{\min_{z\in\overline{\Omega}} S(z)}{Ch}\right\} < \delta \ll h^\kappa.$$

Remark 1.3.3. We chose these hypotheses (cf. Hypothesis 1.3.1 and 1.3.2) because the aim of this section is to treat the two-point eigenvalue interaction in the interior of the pseudospectrum. The two-point interaction close to the pseudospectral boundary remains an interesting open problem.

Continuing, note that we have the splitting

$$\begin{aligned} T_2(\varphi) &= \mathbb{E}\left[\sum_{z\in\sigma(P_h^\delta)} \varphi(z,z)\mathbb{1}_{B(0,R)}\right] + \mathbb{E}\left[\sum_{\substack{z,w\in\sigma(P_h^\delta)\\ z\neq w}} \varphi(z,w)\mathbb{1}_{B(0,R)}\right] \\ &= \int_{\mathbb{C}^2} \varphi(z,z)d\widetilde{\mu}_2(z,z) + \int_{\mathbb{C}^2} \varphi(z,w)d\nu(z,w). \end{aligned}$$

Both terms are positive linear functionals on $\mathscr{C}_0(\Omega^2)$, and thus the above representation by the two measures $\widetilde{\mu}_2$ and ν is well-defined. The measure $\widetilde{\mu}_2$ is supported on the diagonal $D := \{(z,z); z \in \Omega\}$ and is given by the push-forward of μ_1 under the diagonal map $f : \Omega \to D : x \mapsto (x,x)$, i.e. $\widetilde{\mu}_2 = f_*\mu_1$.

The second measure, ν, is called the two-point intensity measure of Ξ and it is supported on $\Omega^2\backslash D$. Their sum naturally yields μ_2, i.e. $\mu_2 = \widetilde{\mu}_2 + \nu$. We see that μ_2 is not absolutely continuous with respect to the Lebesgue measure on $\mathbb{C}^2$. However, this may be the case for the measure ν.

To study the correlation of two points of the spectrum of P_h^δ, we are interested in the two-point intensity measure ν, given by

$$\mathbb{E}\left[\sum_{\substack{z,w\in\sigma(P_h^\delta)\\ z\neq w}} \varphi(z,w)\mathbb{1}_{B(0,R)}\right] = \int_{\mathbb{C}^2} \varphi(z,w)d\nu(z,w). \tag{1.3.3}$$

In particular, we will give an h-asymptotic formula for its Lebesgue density valid at a distance $\gg h^{3/5}$ from the diagonal. For Ω as in (1.3.1) and $C_2 > 0$, we define the set

$$D_h(\Omega, C_2) := \{(z,w) \in \Omega^2;\ |z-w| \le C_2 h^{3/5}\}. \tag{1.3.4}$$

Before we state our main result of this section, recall from (1.2.14) that the direct image $p_*(d\xi\wedge dx)$ of the symplectic volume form $d\xi \wedge dx$ on T^*S^1 is of the form

$$p_*(d\xi\wedge dx) = \sigma(z)L(dz), \tag{1.3.5}$$

where $\sigma(z)$ is smooth.

Theorem 1.3.4. *Let $\Omega \Subset \Sigma$ be as in (1.3.1). Let $\delta > 0$ be as in Hypothesis 1.3.2 with $\kappa > 51/10$. Let ν be the measure defined in (1.3.3) and let $\sigma(z)$ be as in (1.3.5). Then, for $|z-w| \leq 1/C$ with $C > 1$ large enough, there exist smooth functions*

- $\sigma_h(z,w) = \sigma\left(\frac{z+w}{2}\right) + \mathcal{O}(h)$,
- $K(z,w;h) = \sigma_h(z,w)\frac{|z-w|^2}{4h}(1+\mathcal{O}(|z-w|+h^\infty))$,
- $D^\delta(z,w;h) = \frac{\Lambda(z,w)}{(2\pi h)^2(1-\mathrm{e}^{-2K})}\left(1+\mathcal{O}\left(\delta h^{-\frac{8}{5}}\right)\right)+\mathcal{O}\left(\mathrm{e}^{-\frac{D}{h^2}}\right)$, *with*

$$\begin{aligned}\Lambda(z,w;h) =& \sigma_h(z,z)\sigma_h(w,w) + \sigma_h(z,w)^2(1+\mathcal{O}(|z-w|))\mathrm{e}^{-2K}\\ &+ \frac{\sigma_h(z,w)^2(1+\mathcal{O}(|z-w|))}{\mathrm{e}^K\sinh(K)}\left(2K^2\coth(K)-4K\right)\\ &+\mathcal{O}\left(h^\infty+\delta h^{-\frac{31}{10}}\right)\end{aligned}$$

and there exists a constant $c > 0$ such that for all $\varphi \in \mathscr{C}_0^\infty(\Omega^2\backslash D_h(\Omega,c))$ with

$$\int_{\mathbb{C}^2}\varphi(z,w)d\nu(z,w) = \int_{\mathbb{C}^2}\varphi(z,w)D^\delta(z,w;h)L(d(z,w)).$$

Recall from Theorem 1.2.12 that the one-point density of eigenvalues in Ω, as in (1.3.1), is given by

$$\mathbb{E}[\Xi(\varphi)\mathbb{1}_{B(0,R)}] = \int\varphi(z)d(z;h)L(dz), \quad \forall\varphi\in\mathscr{C}_0(\Omega),$$

where

$$d(z;h) = \frac{1}{2\pi h}\sigma(z)+\mathcal{O}(1), \tag{1.3.6}$$

where $\sigma(z)$ is as in (1.3.5). In other words, we know from Theorem 1.2.12 that the average density of eigenvalues in Ω is up to first order determined by symplectic volume form in phase space (we recall that here we only treat the case of Ω being in the interior of the pseudospectrum).

Theorem 1.3.4 agrees very well with this result as that the leading terms to the density $D^\delta(z,w;h)$ (cf. Theorem 1.3.4) are as well determined by symplectic volume form in phase space.

1.3.1 – Interaction

Using the formula obtained in Theorem 1.3.4, we will prove that two eigenvalues of P_h^δ exhibit the following interaction:

Proposition 1.3.5. *Under the hypothesis of Theorem 1.3.4, we have that*

- *for $h^{\frac{4}{7}} \ll |z-w| \ll h^{\frac{1}{2}}$*

$$D^\delta(z,w;h) = \frac{\sigma_h^3(z,w)|z-w|^2}{(4\pi)^2h^3}\left(1+\mathcal{O}\left(\frac{|z-w|^2}{h}+\delta h^{-\frac{8}{10}}\right)\right);$$

- *for $|z-w| \gg (h\ln h^{-1})^{\frac{1}{2}}$*

$$D^\delta(z,w;h) = \frac{\sigma(z)\sigma(w)+\mathcal{O}(h)}{(2h\pi)^2}\left(1+\mathcal{O}\left(\delta h^{-\frac{8}{5}}\right)\right).$$

Let us give some comments on this result: The fact that we cannot analyze the eigenvalue interaction completely up to the diagonal is due to some technical difficulties. In the above proposition, two eigenvalues of the perturbed operator P_h^δ show the following types of interaction:

Short range repulsion The two-point density decays quadratically in $|z-w|$ if two eigenvalues are too close, and we conjecture that this is the case for all z,w as above satisfying $|z-w| \ll h^{\frac{1}{2}}$.

Long range decoupling If the distance between two eigenvalues is $\gg (h\ln h^{-1})^{\frac{2}{3}}$ the two-point density is given by the product of two one-point densities (cf. (1.3.6)). This means that at this distance two eigenvalues are placed in average in an uncorrelated way.

1.3.2 – Conditional density function

We can reformulate Proposition 1.3.5 in terms of the conditional density function: It follows from (1.3.6), (1.2.14) that for $h > 0$ small enough $d(z;h) > 0$ for all $z \in \Omega$ as in (1.3.1). Hence, under the assumptions of Theorem 1.3.4, the conditional average density of eigenvalues of P_h^δ given that $w_0 \in \sigma(P_h^\delta)$ is well defined and given by

$$D_{w_0}^\delta(z;h) := \frac{D^\delta(z, w_0; h)}{d(w_0;h)}.$$

We have the following asymptotic behavior of conditional average density $D_{w_0}^\delta(z;h)$:

Proposition 1.3.6. *Under the hypothesis of Theorem 1.3.4, we have that for $w_0 \in \Omega$*

- *for $h^{\frac{4}{7}} \ll |z - w_0| \ll h^{\frac{1}{2}}$*

$$D_{w_0}^\delta(z;h) = \frac{\sigma_h^2(z)|z-w_0|^2}{8\pi h^2}\left(1 + \mathcal{O}\left(\frac{|z-w_0|^2}{h} + \delta h^{-\frac{8}{5}}\right)\right) \ll 1;$$

- *for $|z - w_0| \gg (h \ln h^{-1})^{\frac{1}{2}}$*

$$D_{w_0}^\delta(z;h) = \frac{\sigma(z) + \mathcal{O}(h)}{2h\pi}\left(1 + \mathcal{O}\left(\delta h^{-\frac{8}{5}}\right)\right).$$

In the above proposition we see that, given an eigenvalue $w_0 \in \sigma(P_h^\delta)$, the density of finding another eigenvalue in the vicinity of w_0 shows the following behavior:

Short range repulsion The density $D_{w_0}^\delta(z;h)$ decays quadratically in $\sigma_h(z)|z - w_0|$ if the distance between z and w_0 is smaller than a term of order $h^{\frac{1}{2}}$. Recall from Proposition 1.2.14 that $\sigma(z)$ grows towards the boundary of Σ, hence the short range repulsion is weaker for Ω close to the boundary of Σ, as we expected from the numerical simulations, see Figure 1.6.

Long range decoupling If the distance between z and w_0 is larger than a term of order $(h \ln h^{-1})^{\frac{1}{2}}$, the density $D_{w_0}^\delta(z;h)$ is given up to a small error by the 1-point density $d(z;h)$ (see (1.3.6)). Hence, we see that at these distances two eigenvalues of P_h^δ are up to a small error uncorrelated.

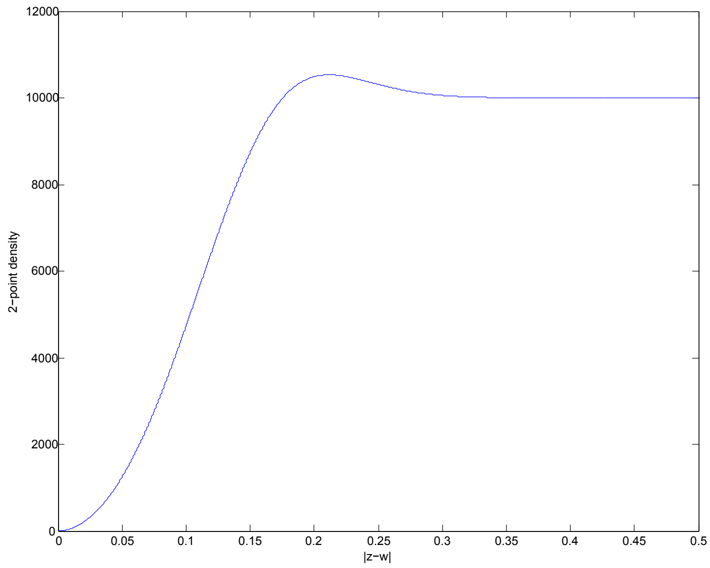

Figure 1.10: Plot of the principal terms of conditional average density $D_{w_0}^\delta(z;h)$ for $w_0 = 0$.

To illustrate Proposition 1.3.6, Figure 1.10 shows a plot of of the principal terms of the conditional density $D_{w_0}^\delta$ as a function of $|z|$, for $w_0 = 0$ and $h = 0.01$, assuming for simplicity that $\sigma(z) = \text{const}$. On the left hand side of the graph we see the quadratic decay, whereas on the right hand side the density is given by $(2\pi h)^{-1}\sigma(z)$.

1.4 | Perturbations of large Jordan blocks

We now turn away from the case of semiclassical differential operators and towards the case of large Jordan matrices. We are interested in the spectrum of a random perturbation of the large Jordan block A_0 :

$$A_0 = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix} : \mathbb{C}^N \to \mathbb{C}^N. \tag{1.4.1}$$

The spectrum of A_0 is

$$\sigma(A_0) = \{0\}.$$

As established in the introduction above, we have that the closed unit disc $\overline{D(0,1)}$ is a zone of spectral instability and if $A_\delta = A_0 + \delta Q$ is a small random perturbation of A_0 we expect the eigenvalues to move inside a small neighborhood of $\overline{D(0,1)}$.

We are interested in the distribution of eigenvalues as the dimension of the matrix gets large, i.e. the limit $N \to \infty$. This situation is inherently different from the above case of semiclassical differential operators since now we are considering here a problem with boundary.

We are interested in the small random perturbations of A_0:

Hypothesis 1.4.1 (Random Perturbation of Jordan block)**.** Let $0 < \delta \ll 1$ and consider the following random perturbation of A_0 as in (1.4.1):

$$A_\delta = A_0 + \delta Q, \quad Q = (q_{j,k}(\omega))_{1\le j,k\le N}, \tag{1.4.2}$$

where $q_{j,k}(\omega)$ are independent and identically distributed complex random variables, following the complex Gaussian law $\mathcal{N}_{\mathbb{C}}(0,1)$.

E.B. Davies and M. Hager [16] studied random perturbations of A_0. They showed that with probability close to 1, most of the eigenvalues are close to a circle:

Theorem 1.4.2 (E.B. Davies-M. Hager [16])**.** *Let A_δ be as in Hypothesis 1.4.1. If $0 < \delta \le N^{-7}$, $R = \delta^{1/N}$, $\sigma > 0$, then with probability $\ge 1 - 2N^{-2}$, we have $\sigma(A_\delta) \subset D(0, RN^{3/N})$ and*

$$\#(\sigma(A_\delta) \cap D(0, Re^{-\sigma})) \le \frac{2}{\sigma} + \frac{4}{\sigma} \ln N.$$

A recent result by A. Guionnet, P. Matched Wood and O. Zeitouni [30] implies that when δ is bounded from above by $N^{-\kappa-1/2}$ for some $\kappa > 0$ and from below by some negative power of N, then

$$\frac{1}{N} \sum_{\mu \in \sigma(A_\delta)} \delta(z-\mu) \to \text{the uniform measure on } S^1,$$

weakly in probability.

Question

Our main focus lies on obtaining, for a small coupling constant δ, more information about the distribution of eigenvalues of A_δ in the interior of a disc, where the result of Davies and Hager only yields a logarithmic upper bound on the number of eigenvalues (see Theorem 1.4.3 below). In particular we are interested in a precise asymptotic formula (as $N \to \infty$) for the density of eigenvalues in this region.

In order to obtain more information in this region, we will study the expected eigenvalue density, adapting the approach of [83]. (For random polynomials and Gaussian analytic functions such results are more classical, [40, 61, 38, 75, 62, 59].)

1.4.1 – Main results on perturbed Jordan block matrices

According to Proposition 1.1.4 we have

$$\mathbb{P}(\|Q\|_{\mathrm{HS}}^2 \geq x) \leq \exp\left(\frac{C_0}{2}N^2 - \frac{x}{2}\right)$$

and hence if $C_1 > 0$ is large enough,

$$\|Q\|_{\mathrm{HS}}^2 \leq C_1^2 N^2, \text{ with probability } \geq 1 - e^{-N^2}. \tag{1.4.3}$$

In particular (1.4.3) holds for the ordinary operator norm of Q. We now state the principal result.

Theorem 1.4.3. *Let A_δ be the $N \times N$-matrix in (1.4.2) and restrict the attention to the parameter range $e^{-N/\mathcal{O}(1)} \leq \delta \ll 1$, $N \gg 1$. Let r_0 belong to a parameter range,*

$$\frac{1}{\mathcal{O}(1)} \leq r_0 \leq 1 - \frac{1}{N},$$

$$\frac{r_0^{N-1}N}{\delta}(1-r_0)^2 + \delta N^3 \ll 1, \tag{1.4.4}$$

so that $\delta \ll N^{-3}$. Then, for all $\varphi \in \mathscr{C}_0(D(0, r_0 - 1/N))$

$$\mathbb{E}\left[\mathbb{1}_{B_{\mathbb{C}^{N^2}}(0,C_1 N)}(Q) \sum_{\lambda \in \sigma(A_\delta)} \varphi(\lambda)\right] = \frac{1}{2\pi}\int \varphi(z)\Xi(z)L(dz),$$

where

$$\Xi(z) = \frac{4}{(1-|z|^2)^2}\left(1 + \mathcal{O}\left(\frac{|z|^{N-1}N}{\delta}(1-|z|)^2 + \delta N^3\right)\right). \tag{1.4.5}$$

is a continuous function independent of r_0. $C_1 > 0$ is the constant in (1.4.3).

Let us give some comments on this result: Theorem 1.4.3 states that the average density of eigenvalues in the disk of radius $r_0 - N^{-1}$ is given by (1.4.5). The result of E.B. Davies and M. Hager [16] (cf. Theorem 1.4.2) only yields a logarithmic upper bound in this region. Conditions $\frac{1}{\mathcal{O}(1)} \leq r_0 \leq 1 - N^{-1}$ and (1.4.4) are needed to restrict the support of the test function φ to the disk inside the pseudospectrum where the average density of eigenvalues is determined by (1.4.5). Outside this disk we obtain no information, however we refer the reader to [63] which treats this case and obtains a probabilistic angular Weyl law in a small neighborhood of the unit circle assuming larger perturbations.

Remark 1.4.4. However, we strongly believe that our methods can be extended to yield a complete average density of eigenvalues in the disk of radius r_0 satisfying $\frac{1}{\mathcal{O}(1)} \leq r_0 \leq 1 - 2/N$, similar as in the case of Hager's model operator (cf. Section 1.2).

Condition (1.4.4) is equivalent to $\delta N^3 \ll 1$ and

$$r_0^{N-1}(1-r_0)^2 \ll \frac{\delta}{N}.$$

For this inequality to be satisfied, it is necessary that

$$r_0 < 1 - 2(N+1)^{-1}.$$

For such r_0 the function $[0, r_0] \ni r \mapsto r^{N-1}(1-r)^2$ is increasing, and so inequality (1.4.4) is preserved if we replace r_0 by $|z| \leq r_0$ and the remainder term in (1.4.5) is small.

The leading contribution to the density $\Xi(z)$ is independent of N and is equal to the Lebesgue density of the volume form induced by the Poincaré metric on the disc $D(0,1)$. This yields a very small density of eigenvalues close to the center of the disc $D(0,1)$ which is, however, growing towards the boundary of $D(0,1)$.

A similar result has been obtained by M. Sodin and B. Tsirelson in [76] for the distribution of zeros of a certain class of random analytic functions with domain $D(0,1)$ linking the fact that the density is given by the volume form induced by the Poincaré metric on $D(0,1)$ to its invariance under the action of $SL_2(\mathbb{R})$.

1.4.2 – Numerical Simulations

To illustrate the result of Theorem 1.4.3, we present the following numerical calculations (Figure 1.11 and 1.12) for the eigenvalues of the $N \times N$-matrix in (1.4.2), where $N = 500$ and the coupling constant δ varies from 10^{-5} to 10^{-2}.

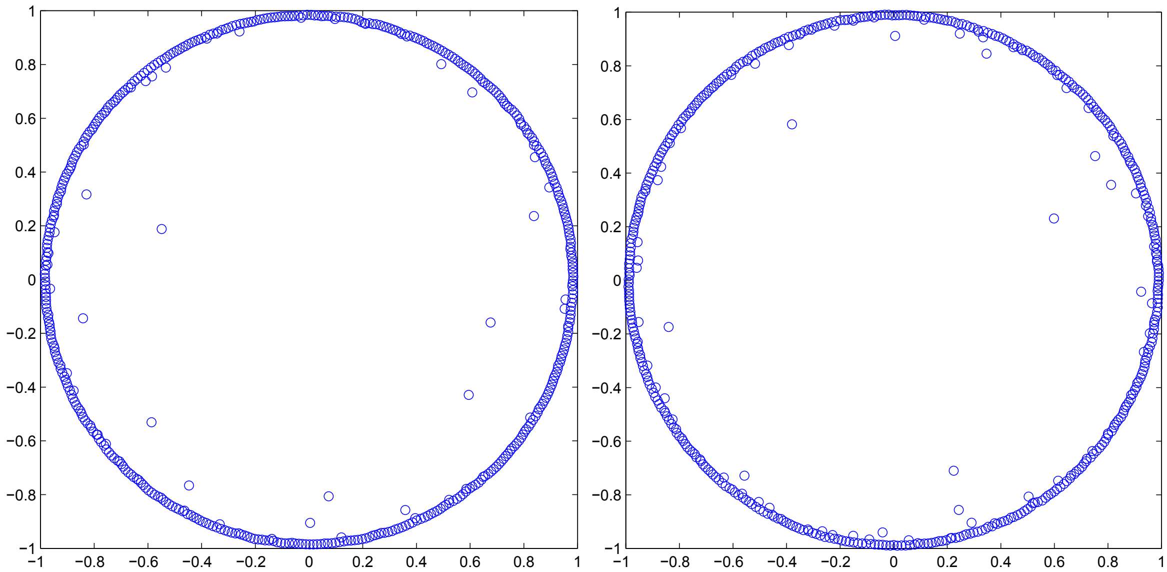

Figure 1.11: On the left hand side $\delta = 10^{-5}$ and on the right hand side $\delta = 10^{-4}$.

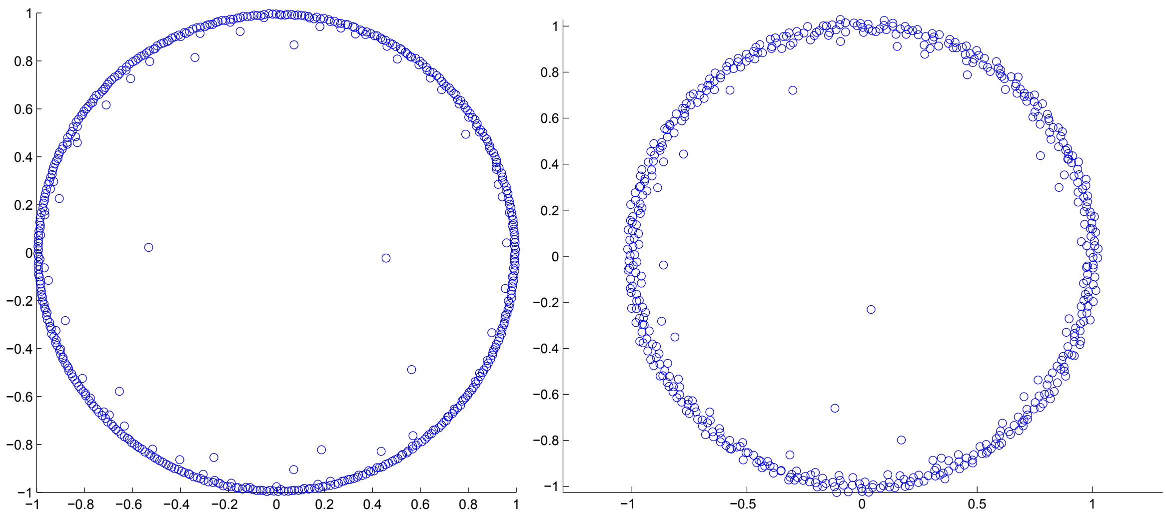

Figure 1.12: On the left hand side $\delta = 10^{-3}$ and on the right hand side $\delta = 10^{-2}$.

In Figure 1.11 and 1.12 we can see that most eigenvalues are in a close vicinity of the unit circle, confirming the results obtained by E.B. Davies and M. Hager [16] (cf Theorem 1.4.2) as well as by A. Guionnet, P. Matched Wood and O. Zeitouni.

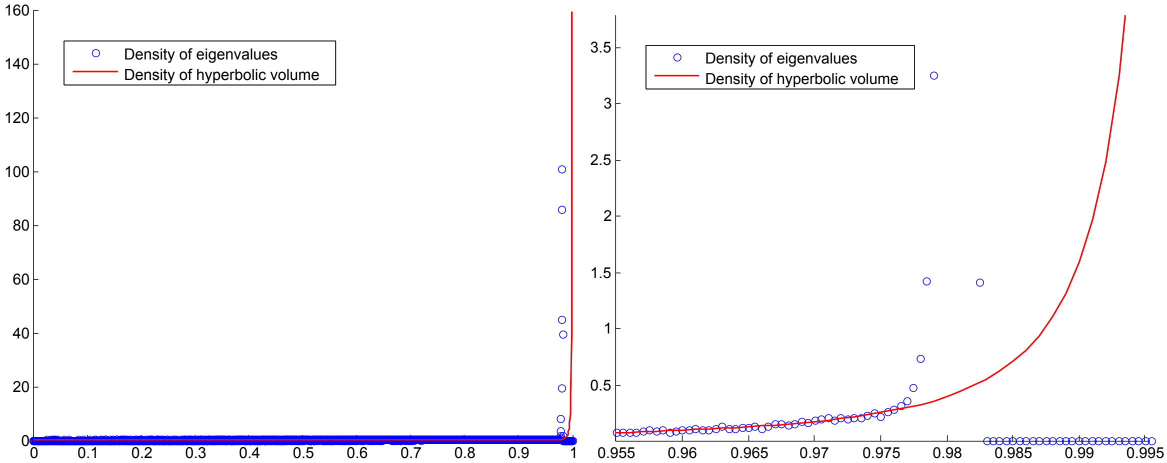

Figure 1.13: The left hand side shows the experimental density of eigenvalues (averaged over 500 realizations), as a function of the radius, of a 1001×1001-Jordan block matrix perturbed with a random complex Gaussian matrix and with coupling $\delta = 2 \cdot 10^{-10}$. The red line is the radial part of density of the hyperbolic volume form on the unit disk. The right hand side presents a magnification of the left hand side, enlarging the zone where the approximation with the hyperbolic volume fails.

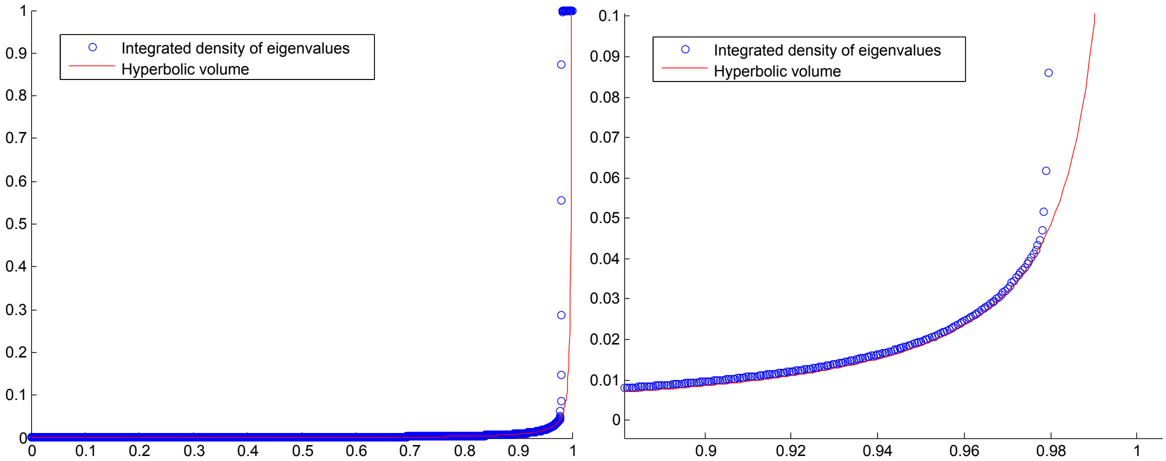

Figure 1.14: The left hand side shows the experimental integrated density of eigenvalues (averaged over 500 realizations), as a function of the radius, of a 1001×1001-Jordan block matrix perturbed with a random complex Gaussian matrix and with coupling $\delta = 2 \cdot 10^{-10}$. The red line is the hyperbolic volume on the unit disk as a function of the radius. The right hand side presents a magnification of the left hand side, enlarging the zone where the approximation with the hyperbolic volume fails.

Furthermore, we can see that the density of eigenvalues in the interior of the unit disc grows towards the boundary of the disc, which is in agreement with the results obtained in Theorem 1.4.3 since the density Ξ (given in 1.4.5) grows towards the boundary.

Figures 1.13 compares the radial part of the density of the hyperbolic volume on the unit disk with the radial experimental (averaged over 500 realizations of random complex Gaussian matrices) density of eigenvalues of a 1001×1001-Jordan block matrix perturbed with a random complex Gaussian matrix with coupling $\delta = 2 \cdot 10^{-10}$. Figures 1.14 shows the same for the respective integrated densities as functions of the radius. These Figures show that the average density and the average integrated density of eigenvalues of (1.4.2) are determined by the hyperbolic volume on the unit disk, as predicted by Theorem 1.4.3. Moreover, they show that here this approximation starts to break down at a radius of $r_0 \approx 0.977$ which is where condition (1.4.4) starts to fail (for the above values of N and δ).

Finally, let us remark that on the right hand side of Figure 1.12 we can see the onset of a different phenomenon discussed in [63]: When the perturbation becomes too strong the spectral band will grow larger since the effects of the random Gaussian matrix will start to dominate over the Jordan block (we refer also to the circular law for the average density of eigenvalues of random complex Gaussian matrices, see for example [79]).

1.5 | Methods and ideas of the proofs

Chapters 2, 3 and 4 present the proofs of our main results and although they are self-contained we will give a short overview over the general strategy of the proofs of our main results (cf Theorems 1.2.12, 1.3.4 and 1.4.3) as a rough road map through the "labyrinth" of estimates.

Let $\mathscr{H}$ denote a complex separable Hilbert space. We are interested in the spectrum of a random perturbation of an operator $P : D(P) \to \mathscr{H}$, of the form

$$P_{\delta,\omega} := P + \delta Q_\omega$$

where $0 < \delta \ll 1$ and Q_ω is a random operator of the form

$$Q_\omega = \sum_{j,k \leq N} \alpha_{j,k}(\omega) e_k^* e_j,$$

where N is sufficiently large, $e_1, e_2, \dots$ is an orthonormal bases of $\mathscr{H}$ and where $e_k^* e_j u = (u|e_k)e_j$, $u \in \mathscr{H}$. Furthermore, $\alpha_{j,k}(\omega) \sim \mathscr{N}_{\mathbb{C}}(0,1)$ are independent and identically distributed Gaussian random variables with expectation 0 and variance 1. To obtain a compact perturbation we restrict the random variables to a large open ball, i.e. we assume that $\alpha \in B(0, CN) \subset \mathbb{C}^{N^2}$, for some constant $C > 1$ large enough.

To obtain an effective description of the spectrum of $P_{\delta,\omega}$ we will set up an auxiliary problem.

Grushin problem We give a short refresher on Grushin problems since they have become an essential tool and they form a key method in the present work. As reviewed in [74], the central idea is to set up an auxiliary problem of the form

$$\begin{pmatrix} P - z & R_- \\ R_+ & 0 \end{pmatrix} : \mathscr{H}_1 \oplus \mathscr{H}_- \longrightarrow \mathscr{H}_2 \oplus \mathscr{H}_+,$$

where $P - z$ is the operator of interest and $R_\pm$ are suitably chosen. We say that the Grushin problem is well-posed if this matrix of operators is bijective. If $\dim \mathscr{H}_- = \dim \mathscr{H}_+ < \infty$, one usually writes

$$\begin{pmatrix} P - z & R_- \\ R_+ & 0 \end{pmatrix}^{-1} = \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{pmatrix}.$$

The key observation, going back to the Shur complement formula or equivalently the Lyapunov-Schmidt bifurcation method, is that the operator $P(z) : \mathscr{H}_1 \to \mathscr{H}_2$ is invertible if and only if the finite dimensional matrix $E_{-+}(z)$ is invertible and when $E_{-+}(z)$ is invertible, we have

$$P^{-1}(z) = E(z) - E_+(z) E_{-+}^{-1}(z) E_-(z).$$

$E_{-+}(z)$ is sometimes called effective Hamiltonian. In the case of the large Jordan block we may take the vectors

$$e_1 := (1, 0, \dots, 0)^{\mathrm{t}} \in \mathbb{C}^N, \quad e_N := (0, \dots, 0, 1)^{\mathrm{t}} \in \mathbb{C}^N,$$

and set $R_+ u = (u|e_1)$ and $R_- u- = u_- e_N$ (cf Section 4.2 and [74]) to gain a well-posed Grushin problem. In the case of Hager's model operators will use quasimodes, see the paragraph entitled "quasimodes" below.

Grushin problem for the perturbed operator For $\delta > 0$ small enough, we can use the same $R_\pm$ as for the unperturbed operator P, to gain a well-posed Grushin problem for the perturbed operator

$$\begin{pmatrix} P_{\delta,\omega} - z & R_- \\ R_+ & 0 \end{pmatrix} : \mathscr{H}_1 \oplus \mathscr{H}_- \longrightarrow \mathscr{H}_2 \oplus \mathscr{H}_+,$$

with

$$\begin{pmatrix} P_{\delta,\omega} - z & R_- \\ R_+ & 0 \end{pmatrix}^{-1} = \begin{pmatrix} E^{\delta,\omega}(z) & E_+^{\delta,\omega}(z) \\ E_-^{\delta,\omega}(z) & E_{-+}^{\delta,\omega}(z) \end{pmatrix}.$$

Using $E_{-+}^{\delta,\omega}(z)$, we have an effective description of the spectrum of $P_{\delta,\omega}$. In our case $\dim \mathscr{H}_\pm = 1$ (cf Section 2.2 and 4.2), wherefore

$$\sigma(P_{\delta,\omega}) = (E_{-+}^{\delta,\omega})^{-1}(0).$$

Quasimodes For Hager's operator P (cf Section 1.1.1), we will use quasimodes $e_\pm$ for the unperturbed operator P and its adjoint P^* to construct the auxiliary operators $R_\pm$ by setting $R_+u = (u|e_+)$ and $R_-u- = u_-e_-$ (details will be given in Section 2.1 and 2.2). For $e_\pm$, we will use two kinds of quasimodes:

- The eigenfunctions e_0 and f_0 of the self-adjoint auxiliary operators $Q(z)$ and $\widetilde{Q}(z)$ (cf Section 1.2.2), which have the advantage of being valid in all of Σ, see (1.1.13), however, at the price of being less explicit.
- Local WKB approximate solutions e_{wkb} and f_{wkb} of the form

  $$e_{wkb}(x,z;h) = a(z;h)\chi_e(x,z,h)\mathrm{e}^{\frac{i}{h}\phi_+(x,z)}, \quad f_{wkb}(x,z;h) = b(z;h)\chi_f(x,z,h)\mathrm{e}^{\frac{i}{h}\phi_-(x,z)},$$

  where $\phi_\pm(x,z)$ are phases satisfying the eikonal equations

  $$p(x,\partial_x\phi_+) = z, \quad \text{and} \quad \overline{p}(x,\partial_x\phi_-) = \overline{z},$$

  where p is the semiclassical principal symbol of P and $\overline{p}$ the one of P^*. Furthermore, $\chi_{e,f}(x,z,h)$ are smooth compactly supported cut-off functions and $a(z;h) \sim h^{-1/4}(a_0(z) + ha_1(z) + \dots)$ and $b(z;h) \sim h^{-1/4}(b_0(z) + hb_1(z) + \dots)$ are normalization factors.

  These quasimodes are more explicit than e_0 and f_0, they are, however, only valid in certain subsets of Σ.

Moments of linear statistics Using the effective Hamiltonian $E_{-+}^{\delta,\omega}(z)$ of the perturbed operator, we will study the first two moments of linear statistics of the random point process

$$\Xi := \sum_{z\in\sigma(P_{\delta,\omega})} \delta_z = \sum_{z\in(E_{-+}^{\delta,\omega})^{-1}(0)} \delta_z.$$

More precisely, we will study μ_1 the one-point intensity measure of Ξ, given by

$$\mathbb{E}\left[\sum_{z\in(E_{-+}^{\delta,\omega})^{-1}(0)} \varphi(z)\, \mathbb{1}_{B(0,CN)}(\alpha)\right] = \int_{\mathbb{C}} \varphi(z) d\mu_1(z)$$

where φ is a continuous compactly supported function. Moreover, we will study ν, the two-point intensity measure of Ξ, given by

$$\mathbb{E}\left[\sum_{\substack{z,w\in(E_{-+}^{\delta,\omega})^{-1}(0)\\ z\neq w}} \varphi(z,w)\, \mathbb{1}_{B(0,CN)}(\alpha)\right] = \int_{\mathbb{C}^2} \varphi(z,w) d\nu(z,w).$$

In particular, we will examine their Lebesgue densities and use these to obtain Theorems 1.2.12, 1.3.4 and 1.4.3 and their consequences.

There are two essential steps involved in obtaining these densities:

1. We obtain a formula to describe these densities.

   (a) In the case of the Jordan block, $E_{-+}^{\delta,\omega}$ depends holomorphically on z and on the random variables α which is very useful. We will see that

   $$T := \left\{(z,\alpha) \in \Omega \times B(0,CN);\, E_{-+}^{\delta,\omega}(z,\alpha) = 0\right\}$$

   is a a smooth complex hypersurface in $\Omega \times B(0,CN) \subset \mathbb{C} \times \mathbb{C}^{N^2}$, where $\Omega \subset \mathbb{C}$ is open, bounded and connected. Exploiting this, we will show that

   $$\mathbb{E}\left[\sum_{z\in(E_{-+}^{\delta})^{-1}(0)} \varphi(z)\, \mathbb{1}_{B(0,CN)}(\alpha)\right] = \int_T \varphi(z)\mathrm{e}^{-\alpha^*\alpha}(2i)^{-N^2} d\overline{\alpha}\wedge d\alpha, \tag{1.5.1}$$

   where we view $(2i)^{-N^2} d\overline{\alpha}\wedge d\alpha$ as a complex (N^2,N^2)-form on $\Omega \times B(0,CN)$, restricted to T, which yields a non-negative differential form of maximal degree on T.

   (b) In the case of Hager's model operator, $E_{-+}^{\delta,\omega}$ depends only smoothly on z but it satisfies additionally a $\overline{\partial}$-equation, i.e. there exists a smooth function f^δ such that

   $$\partial_{\overline{z}}E_{-+}^{\delta}(z) + f^{\delta}(z)E_{-+}^{\delta}(z) = 0.$$

   Using this, together with approximations of the delta function, we obtain an explicit formula for the one-point density:

   $$\mathbb{E}\left[\sum_{z\in(E_{-+}^{\delta})^{-1}(0)} \varphi(z)\, \mathbb{1}_{B(0,CN)}(\alpha)\right] = \lim_{\varepsilon\to 0}\int \varphi(z)D_\varepsilon(z;h,\delta)L(dz),$$
   $$\text{where } D_\varepsilon(z;h,\delta) := \pi^{-N}\int_{B(0,CN)} \chi\left(\frac{E_{-+}^{\delta}(z,\alpha)}{\varepsilon}\right)\frac{1}{\varepsilon^2}\left|\partial_z E_{-+}^{\delta}(z,\alpha)\right|^2 \mathrm{e}^{-\alpha\overline{\alpha}}L(d\alpha). \tag{1.5.2}$$

   Here, $\chi \in \mathscr{C}_0^\infty(\mathbb{C})$ such that $\chi \geq 0$ and $\int \chi(w)L(dw) = 1$. The formula for the two-point density is similar.

2. The second step to analyze these densities will be to choose appropriate coordinates in the space of random variables: In the case of the one-point densities, we will find a vector $X(z) \in \mathbb{C}^{N^2}$ such that $\|X(z)\| \neq 0$ and

   $$E_{-+}^{\delta,\omega}(z,\alpha) = 0 \Rightarrow \left(\overline{X}(z)\cdot\partial_\alpha\right)E_{-+}^{\delta,\omega}(z,\alpha) \neq 0.$$

   Using this vector we have the following corresponding orthogonal decomposition

   $$\alpha = \beta_1\overline{X}(z) + \beta', \quad \beta' \in \overline{X}(z)^{\perp},\ \beta_1 \in \mathbb{C}.$$

   Here, $\overline{X}(z)^{\perp}$ is identified with $\mathbb{C}^{N^2-1}$ via an orthonormal basis. Performing a change of variables corresponding to this choice of basis in the integrals (1.5.1) and (1.5.2), we will obtain (after a lengthy calculation) the desired asymptotic formulas describing the densities.

   The case of the two-point density is similar.

1.6 | Some open problems

We end the introduction by discussing some interesting open problems on which we are currently working.

1.6.1 – Random perturbations of non-self-adjoint semiclassical pseudo-differential operators

We have seen above some consequences of random perturbations on the spectra of non-self-adjoint operators. However, there are many more compelling open questions.

Generalizations of the results The methods used to prove the result on Hager's model can be extended to a much broader class of one-dimensional semiclassical pseudo-differential operators. It would also be very interesting to consider the case of small multiplicative random perturbations of differential operators since these allow us to remain in the class of differential operators.

Furthermore, to obtain similar results on the average density in all of the pseudo-spectrum would be very interesting in the case of multi-dimensional semiclassical pseudo-differential operators.

The Jordan block matrix can be see as a model for a differential operator with boundary conditions. We have seen that in this case eigenvalues are produced through small random perturbations even outside the image of the principal symbol. Further investigating this phenomenon seems very promising.

Interaction close to the pseudospectral boundary In the above we have only given a description of the interaction of two eigenvalues in the interior of the pseudospectrum. However, we still miss a description of the interaction of two eigenvalues close to the pseudospectral boundary. In view of the numerical simulations presented in Figure 1.1 and of Theorem 1.2.12 it is clear that the behavior of the eigenvalues changes completely when approaching the pseudospectral boundary.

Weaker non-self-adjointness The class of semiclassical differential operators that we considered in this thesis (cf Section 1.1.1) has the property that the semiclassical principal symbol p (cf (1.1.7)) is complex valued. However, in the case of the damped wave equation (cf [64]) the principal symbol is real-valued and the non-self-adjointness comes from the subprincipal symbol. The effects of random perturbations in this case are as of yet unknown.

1.6.2 – Resonances of random Schrödinger operators

Following the discussion on resonances of Schrödinger operators at the beginning of this chapter, we turn now to the particular case of discrete random Schrödinger operators. Here, the particle is restricted to move on the lattice $\mathbb{Z}^d$ instead of the space $\mathbb{R}^d$. More precisely, we consider the random discrete Anderson model, introduced by P.W. Anderson [3], that is, on $\ell^2(\mathbb{Z}^d)$,

$$H_\omega = -\Delta + \lambda V_\omega,$$

where $-\Delta$ is the free discrete centered Laplace operator given by

$$(-\Delta u)(n) = \sum_{|m-n|=1} u(m), \quad \text{for } u \in \ell^2(\mathbb{Z}^d),$$

and V_ω is a random potential

$$(V_\omega u)(n) = V_\omega(n)u(n), \quad \text{for } u \in \ell^2(\mathbb{Z}^d),$$

and $\lambda > 0$ the coupling constant. We assume that the random variables $(V_\omega)_{n\in\mathbb{Z}^d}$ are independent identically distributed and that their common law admits a bounded compactly supported continuous density g.

Properties of the Anderson model The spectral theory of the Anderson model (and many other types of random Schrödinger operators) has been study extensively, see for example [22, 81, 35, 10, 28, 49, 26, 46, 57] and the references in [42].

Let $\sigma(H_\omega)$ be the spectrum of H_ω. It is known (see e.g. [22]) that, ω-almost surely,

$$\sigma(H_\omega) = \Sigma := [-2d, 2d] + \operatorname{supp} g. \tag{1.6.1}$$

The Anderson model satisfies the following important hypotheses:

Wegner estimate (W) Let $I \Subset \Sigma$ be a relatively compact open subset of the almost sure spectrum Σ. We say that a Wegner estimate hold in I, if there exists a $C > 0$ such that, for $J \subset I$, and a cube $\Lambda \subset \mathbb{Z}^d$, one has

$$\mathbb{E}\left[\operatorname{tr}(\mathbb{1}_J(H_\omega(\Lambda))\right] \le C|J||\Lambda|. \tag{1.6.2}$$

Here, $\mathbb{E}[\cdot]$ denotes the expectation with respect to the random variables and $H_\omega(\Lambda)$ denotes the operator H_ω restricted to the cube $\Lambda \subset \mathbb{Z}^d$ with Dirichlet boundary conditions (other boundary conditions work as well, e.g. periodic boundary conditions). More precisely, for $L \ge 1$, $\Lambda_L = \Lambda$ denotes the cube $[\![-L, L]\!]^d := [-L, L]^d \cap \mathbb{Z}^d \subset \mathbb{Z}^d$. In the sequel we will write $|\Lambda| \to \infty$, meaning that $L \to \infty$.

A Wegner estimate has been proven for many different random Schrödinger operators, such as the Anderson model, both in the discrete and the continuous case under quite general conditions on potentials and randomness, see for example [81, 35, 10, 28]. The left hand side of (1.6.2) yields an upper bound on the probability to have at least one eigenvalue of the operator $H_\omega(\Lambda)$ in J.

By **(W)**, we have that the integrated density of states, defined by

$$N(E) := \lim_{|\Lambda| \to \infty} \frac{\#\{\lambda \in \sigma(H_\omega(\Lambda));\ \lambda \le E\}}{|\Lambda|},$$

is the distribution function of a measure that is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$. We denote by $\mathbb{R} \ni E \mapsto n(E)$, defined E-almost everywhere, the density of states which is the Lebesgue density of the above measure. Furthermore, for any continuous function $\varphi : \mathbb{R} \to \mathbb{R}$, we have that

$$\int_{\mathbb{R}} \varphi(E) n(E) dE = \mathbb{E}[\langle \delta_0, \varphi(H_\omega)\delta_0 \rangle].$$

Here, $\delta_i \in \ell^2(\mathbb{Z}^d)$ is defined by $\delta_i(j) = 0$ for $i \ne j$ and $\delta_i(j) = 1$ for $i = j$. In fact the collection $\{\delta_i\}_{i \in \mathbb{Z}^d}$ is an orthonormal basis of $\ell^2(\mathbb{Z}^d)$.

Another important consequence of **(W)** is that any given $E \in J$ is not an eigenvalues of $\sigma(H_\omega(\Lambda))$ for almost all ω.

Minami estimate (M) Let $I \Subset \Sigma$ be a relatively compact open subset of the almost sure spectrum Σ. We say that a Minami estimate hold in I, if there exists a $C > 0$ such that, for $J \subset I$, and a cube $\Lambda \subset \mathbb{Z}^d$, one has

$$\mathbb{E}\left[\operatorname{tr}(\mathbb{1}_J(H_\omega(\Lambda))\left[\operatorname{tr}(\mathbb{1}_J(H_\omega(\Lambda)) - 1\right]\right] \le C(|J||\Lambda|)^2. \tag{1.6.3}$$

The Minami estimate is proven for much less models than the Wegner estimate. However, in the case of the discrete Anderson model it has been proven to hold for $I = \Sigma$, see [49]. The right hand side can be lower bound by the probability to find at least two eigenvalues in J. The Minami estimate tells us that the eigenvalues of $H_\omega(\Lambda)$ are ω-almost surely simple.

Localization (Loc) Let $I \subset \Sigma$ be a compact interval. We say that I lies in the region of complete localization if for all $\xi \in]0, 1[$, we have

$$\sup_{L > 0} \sup_{\substack{\operatorname{supp} f \subset I \\ |f| \le 1}} \mathbb{E}\left(\sum_{\gamma \in \mathbb{Z}^d} e^{|\gamma|^\xi} \left\| \mathbb{1}_{\{0\}} f(H_\omega(\Lambda)) \mathbb{1}_{\{\gamma\}} \right\|_2 \right) < \infty. \tag{1.6.4}$$

Here, f is a Borel function on $\mathbb{R}$.

We note that **(Loc)** implies that the spectrum of H_ω is pure point in I (cf [42, 27]) with associated sub-exponentially decaying eigenfunctions. It is known that there exists a λ_0 such that for all $\lambda \geq \lambda_0$ we have that **(Loc)** holds for all $I \subset \Sigma$ (cf [1]).

In case of the discrete Anderson model we have the finite volume fractional moment method available. For I satisfying the finite volume fractional moment criteria (cf [2]) for large enough cubes Λ, we may replace $\mathrm{e}^{|\gamma|^\xi}$ in (1.6.4) by $\mathrm{e}^{\eta|\gamma|}$ with $\eta > 0$. In particular for large enough coupling λ we have this for $I = \Sigma$, for large enough cubes Λ, with associated exponentially decaying eigenfunctions (cf [44, 1]):

There exists $\nu(\lambda) > 0$ such that, for any $p > 0$, there exists $q > 0$ and $L_0 > 0$ such that, for $L \geq L_0$, with probability $\geq 1 - L^{-p}$, if

(1) $\varphi_{n,\omega}$ is a normalized eigenvector of $H_\omega(\Lambda)$ associated to an eigenvalue $E_{n,\omega}(\Lambda) \in \Sigma$,

(2) $x_{n,\omega} \in \Lambda$ is a maximum of $x \to |\varphi_{n,\omega}(x)|$ in Λ,

then, for $x \in \Lambda$, one has

$$|\varphi_{n,\omega}(x)| \leq L^q \mathrm{e}^{-\nu(\lambda)|x - x_{n,\omega}|}. \tag{1.6.5}$$

Here, the point $x_{n,\omega}$ is called a localization center for $\varphi_{n,\omega}$.

Resonances for a random potential restricted to a large box The main object of interest is the self-adjoint operator

$$H_{\omega,\Lambda} := -\Delta + \lambda V_\omega \chi_\Lambda \tag{1.6.6}$$

as $|\Lambda| \to \infty$. Here, $\chi_\Lambda(n) = 1$ if $n \in \Lambda$ and 0 if not.

Since $V_\omega \chi_\Lambda$ is compact and self-adjoint, it follows from Weyl's essential spectrum theorem (cf for example [55]) that the essential spectrum of $H_{\omega,\Lambda}$ is that of $-\Delta$, that is $[-2d, 2d]$. The operator $H_{\omega,\Lambda}$ has therefore only discrete spectrum in $\mathbb{R} \backslash [-2d, 2d]$.

We are interested in giving a description of the resonances of the operator close to the real axis. These can be defined as the poles of the meromorphic continuation of the resolvent of $H_{\omega,\Lambda}$ through $]-2d, 2d[$.

Meromorphic continuation of the resolvent By the discrete Fourier transformation $\mathscr{F} : \ell^2(\mathbb{Z}^d) \to L^2(\mathbb{R}^d/(2\pi\mathbb{Z}^d))$, we see that H_0 is a Fourier multiplier with symbol

$$p(\theta) := 2\sum_{k=1}^{d} \cos\theta_k \in [-2d, 2d] =: \mathbb{T}^d. \tag{1.6.7}$$

p is a Morse function with critical values given by

$$\Lambda_0 := \{-2d + 4k;\ 0 \leq k \leq d\}. \tag{1.6.8}$$

Using (1.6.7), one has, for $H_0 := -\Delta$, for $\operatorname{Im} z > 0$ and for $n, m \in \mathbb{Z}^d$, that the kernel of $R_0(z)$, the resolvent of H_0, is given by

$$R_0(z; n, m) := \langle (H_0 - z)^{-1}\delta_m | \delta_n \rangle = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \frac{\mathrm{e}^{i(n-m)\theta}}{p(\theta) - z} d\theta. \tag{1.6.9}$$

We are interested in the analytical continuation of (1.6.9) from $\mathbb{C}^+$ when z crosses through $]-2d, 2d[$. Analogous integrals have already been studied extensively, see e.g. [50, 23, 43, 45], and one can prove the following result.

Theorem 1.6.1. *The operator valued function $\mathbb{C}^+ \ni z \mapsto (H_0 - z)^{-1}$ admits an analytic continuation form $\mathbb{C}^+$ to*

$$\mathbb{C} \backslash \left( ]-\infty, -2d] \cup \bigcup_{1 \leq k \leq d-1} (-2d + 4k - i\mathbb{R}_+) \cup [2d, \infty[\right)$$

with values in the operators from $\ell^2_{comp}(\mathbb{Z}^d)$ to $\ell^2_{loc}(\mathbb{Z}^d)$.

Using analytic Fredholm theory, one deduces from Theorem 1.6.1 the following result.

Theorem 1.6.2. *The operator valued function* $\mathbb{C}^+ \ni z \mapsto R_{\omega,\Lambda}(z) := (H_{\omega,\Lambda} - z)^{-1}$ *admits a meromorphic continuation form* $\mathbb{C}^+$ *to*

$$\mathbb{C} \setminus \left( ]-\infty, -2d] \cup \bigcup_{1\le k \le d-1} (-2d + 4k - i\mathbb{R}_+) \cup [2d, \infty[\right)$$

with values in the operators from $\ell^2_{comp}(\mathbb{Z}^d)$ *to* $\ell^2_{loc}(\mathbb{Z}^d)$.

The resonances are defined as the poles of this meromorphic continuation, see Figure 1.15.

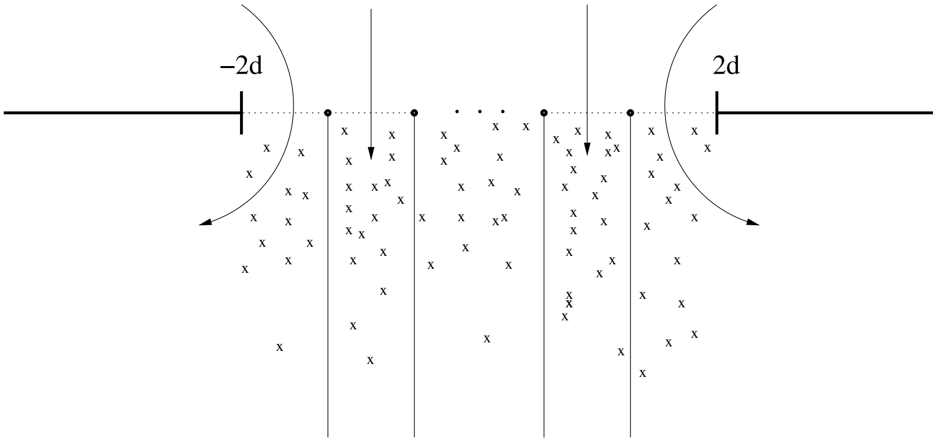

Figure 1.15: Resonances as poles of the meromorphic continuation of the resolvent $(H_{\omega,\Lambda} - z)^{-1}$.

The case of $d = 1$ has been studied extensively by F. Klopp, see [45]. Therein, Klopp gives a detailed description of the resonances of $H_{\omega,\Lambda}$ and compares them to the case of resonances of periodic Schrödinger operators (i.e. the potential V is periodic and not random). He proves that in both cases there is a gap between the real axis and the resonances. However, remarkably, in the random case the width of this gap is exponentially small in L, whereas in the periodic case it is only polynomially small in L.

Theorem 1.6.3 (F. Klopp [45])**.** *Let* $d = 1$ *and let* I *be a compact interval in* $]-2,2[\cap\mathring{\Sigma}$ *(cf (1.6.1)). Then,* ω*-almost surely, one has that for* $\varepsilon \in]0,1[$*, there exists* $L_0 > 0$ *such that, for* $L \ge L_0$*, there are no resonances of* $H_{\omega,\Lambda}$ *in the rectangle*

$$\left\{ z \in \mathbb{C};\ \mathrm{Re}\, z \in I,\ \mathrm{Im}\, z \ge -\mathrm{e}^{-\rho L(1+\varepsilon)} \right\}$$

where ρ *is the maximum of the Lyapunov exponent* $\rho(E)$ *on* I.

We recall that the Lyapunov exponent $\rho(E)$ is defined as follows.

$$\rho(E) := \lim_{L\to\infty} \frac{\ln \| T_L(E,\omega) \|}{L+1},$$

where $T_L(E,\omega)$ is the L-step transfermatrix, i.e.

$$T_L(E,\omega) := \begin{pmatrix} E - V_\omega(L) & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} E - V_\omega(0) & -1 \\ 1 & 0 \end{pmatrix}.$$

The number resonances of $H_{\omega,\Lambda}$ closest to the real axis is given asymptotically by the integrated density of states. Indeed, F. Klopp proves in [45] the following result.

Theorem 1.6.4 (F. Klopp [45])**.** *Let* $d = 1$ *and let* I *be a compact interval in* $]-2,2[\cap\mathring{\Sigma}$*. Then, for any* $\kappa \in]0,1[$*,* ω*-almost surely, one has*

$$\frac{\#\left\{ z \text{ resonance of } H_{\omega,\Lambda} \text{ s.t } \mathrm{Re}\, z \in I,\ \mathrm{Im}\, z \le -\mathrm{e}^{-L^\kappa} \right\}}{L} \longrightarrow \int_I n(E)\, dE,\ L \to \infty.$$

To prove this result Klopp uses the eigenvectors $\varphi_{n,\omega}$ associated to the energies $E_{n,\omega}$ (cf (1.6.5)) as quasimodes for the operator $H_{\omega,\Lambda}$ to construct resonances (we refer also to the similar works [78, 77]).

Using **(M)** and **(Loc)**, for a large enough coupling constant λ, we have exponentially decaying eigenvectors $\varphi_{n,\omega}$ associated to almost surely simple energies $E_{n,\omega}$. This should allow us to follow a strategy similar to Klopp's to prove the extension of Theorem 1.6.4 to d-dimensions. Due to some preliminary results we strongly believe that Theorem 1.6.4 holds true in the d-dimensional case.

Conjecture 1.6.5. *Let $I \subset]-2d, 2d[\cap \mathring{\Sigma}$ be a compact interval. Then, for some constant $C \gg 1$, ω-almost surely, one has*

$$\frac{\#\left\{z \textit{ resonance of } H_{\omega,\Lambda} \textit{ s.t} \operatorname{Re} z \in I,\ \operatorname{Im} z \leq -\mathrm{e}^{-CL}\right\}}{L} \longrightarrow \int_I n(E)dE,\ L \to \infty.$$

CHAPTER 2

AVERAGE DENSITY OF EIGENVALUES FOR A CLASS OF NON-SELF-ADJOINT OPERATORS UNDER RANDOM PERTURBATIONS

The intention of this chapter is to prove the results discussed in Section 1.2. We consider Hager's model operator (cf (1.1.9)), a non-self-adjoint h-differential model operator P_h in the semiclassical limit ($h \to 0$), subject to small random perturbations.

We study the intensity measure of the random point process of eigenvalues and prove an h-asymptotic formula for the average density of eigenvalues. With this we show that there are three distinct regions of different spectral behavior in Σ: The interior of the pseudospectrum is solely governed by a Weyl law, close to its boundary there is a strong spectral accumulation given by a tunneling effect followed by a region where the density decays rapidly. The material presented in this chapter can be found in [83].

2.1 | Quasimodes

The purpose of this section is to construct quasimodes for the operator

$$P_h - z$$

for $z \in \Omega \Subset \Sigma$ with

$$\Omega \Subset \Sigma \text{ is open, relatively compact with } \operatorname{dist}(\Omega, \partial\Sigma) > Ch^{2/3} \text{ for some constant } C > 0. \qquad (2.1.1)$$

We will in particular always assume that this assumption on $\Omega \Subset \Sigma$ is satisfied, if nothing else is specified.

We make the distinction between the following two cases:

Quasimodes in the interior of Σ We consider z being in the *interior* of Σ, i.e. $z \in \Omega_i \Subset \mathring{\Sigma}$ such that there exists a constant $C_{\Omega_i} > 0$ such that

$$\operatorname{dist}(\Omega_i, \partial\Sigma) > \frac{1}{C_{\Omega_i}}.$$

In this case, following the approach of Hager [32], we can find quasimodes by a WKB construction for the operator $(P_h - z)$;

Quasimodes close to the boundary Σ We consider z being *close* to the boundary of Σ, i.e. $z \in \Omega \cap (\Omega_\eta^a \cup \Omega_\eta^b)$ where, following the notation used in [4], we define for some constant $C > 0$

$$\begin{aligned} \Omega_\eta^a &:= \left\{ z \in \mathbb{C} : \frac{\eta}{C} \le \operatorname{Im} z \le C\eta \right\}, \\ \Omega_\eta^b &:= \left\{ z \in \mathbb{C} : \frac{\eta}{C} \le (\operatorname{Im} g(b) - \operatorname{Im} z) \le C\eta \right\}, \end{aligned} \tag{2.1.2}$$

with $h^{2/3} \ll \eta \le$ const. (recall from Hypothesis 1.1.2 that $\operatorname{Im} g(a) = 0$). The precise value of the above constant $C > 0$ is not important for the obtained asymptotic results. We will only consider the case $z \in \Omega_\eta^a$ since $z \in \Omega_\eta^b$ can be treated the same way. We may follow the approach of Bordeaux-Montrieux [4] and find quasimodes by a WKB construction for the rescaled operator

$$\widetilde{P}_{\widetilde{h}} - \widetilde{z} := \frac{h}{\eta^{3/2}} D_{\widetilde{x}} + \frac{g(\sqrt{\eta}\widetilde{x})}{\eta} - \frac{z}{\eta} := \widetilde{h} D_{\widetilde{x}} + \widetilde{g}(\widetilde{x}) - \widetilde{z}, \tag{2.1.3}$$

with the rescaling

$$S^1 \ni x = \sqrt{\eta}\widetilde{x} \quad \text{and} \quad \widetilde{h} := \frac{h}{\eta^{3/2}}.$$

Note that in this case demanding $\widetilde{h} \ll 1$ implies the condition $h^{2/3} \ll \eta$. The rescaling is motivated by analyzing the Taylor expansion of $\operatorname{Im} g(x)$ around the critical point a yielding that for $\operatorname{Im} z \to 0$

$$|x_\pm(z) - a| \asymp \sqrt{\eta}, \tag{2.1.4}$$

where $x_\pm(z)$ are as (1.1.14). This shows that the rescaling shifts the problem of constructing quasimodes for z close to the boundary of Σ to constructing quasimodes for z well in the interior of the range of the semiclassical principal symbol of the new operator $\widetilde{P}_{\widetilde{h}}$.

Remark 2.1.1. Throughout this text we shall work with the convention that when writing an estimate, e.g. $\mathcal{O}(\delta^q \eta^r h^s)$ or $A \asymp \eta^r h^s$, we implicitly set $\eta = 1$ when $\operatorname{dist}(z, \partial\Sigma) > 1/C$ but keep η when $z \in \Omega_\eta^a$.

Let us note, that by Taylor expansion we may deduce that $S = S(z)$, as defined in Definition 1.2.2, satisfies

$$S(z) \asymp \eta^{3/2} \tag{2.1.5}$$

2.1.1 – Quasimodes for the interior of Σ

Definition 2.1.2. Let $z \in \Omega_i \Subset \mathring{\Sigma}$ and let x_-, x_+ be as in the introduction. Let $\psi \in \mathscr{C}_0^\infty(\mathbb{R})$ with $\operatorname{supp} \psi \subset]0,1[$ and $\int \psi(x) dx = 1$. Define $\chi_e \in \mathscr{C}_0^\infty(]x_- - 2\pi, x_-[)$ and $\chi_f \in \mathscr{C}_0^\infty(]x_+, x_+ + 2\pi[)$ by

$$\begin{aligned} \chi_e(x,z;h) &:= \int_{-\infty}^x h^{-\frac{1}{2}} \left\{ \psi\left(\frac{y - x_- + 2\pi}{\sqrt{h}}\right) - \psi\left(\frac{x_- - y}{\sqrt{h}}\right) \right\} dy, \\ \chi_f(x,z;h) &:= \int_{-\infty}^x h^{-\frac{1}{2}} \left\{ \psi\left(\frac{y - x_+}{\sqrt{h}}\right) - \psi\left(\frac{x_+ + 2\pi - y}{\sqrt{h}}\right) \right\} dy. \end{aligned} \tag{2.1.6}$$

Furthermore, define for $x \in]x_- - 2\pi, x_-[$

$$\phi_+(x,z) := \int_{x_+}^x \big(z - g(y)\big)\, dy,$$

and for $x \in]x_+, x_+ + 2\pi[$

$$\phi_-(x,z) := \int_{x_-}^x \overline{\big(z - g(y)\big)} dy.$$

Consider the $L^2(S^1)$-normalized quasimodes

$$e_{wkb}(x,z;h) := h^{-\frac{1}{4}} a(z;h)\chi_e(x,z;h)\mathrm{e}^{\frac{i}{h}\phi_+(x,z)} \in \mathscr{C}_0^\infty(]x_- - 2\pi, x_-[) \tag{2.1.7}$$

and

$$f_{wkb}(x,z;h) := h^{-\frac{1}{4}} b(z;h)\chi_f(x,z;h)\mathrm{e}^{\frac{i}{h}\phi_-(x,z)} \in \mathscr{C}_0^\infty(]x_+, x_+ + 2\pi[) \tag{2.1.8}$$

where $a(z;h)$ and $b(z;h)$ are normalization factors obtained by the stationary phase method. Thus, $a(z;h) \sim a_0(z) + ha_1(z) + \dots \neq 0$ and $b(z;h) \sim b_0(z) + hb_1(z) + \dots \neq 0$ depend smoothly on z such that all derivatives with respect to z and $\overline{z}$ are bounded when $h \to 0$.

The quasimodes e_{wkb} and f_{wkb} are WKB approximate null solutions to $(P_h - z)$ and $(P_h - z)^*$ since locally

$$(P_h - z)\mathrm{e}^{\frac{i}{h}\phi_+(x,z)} = 0, \quad \text{and} \quad (P_h - z)^* \mathrm{e}^{\frac{i}{h}\phi_-(x,z)} = 0.$$

This follows from the fact that $\phi_\pm(x,z)$ satisfy the eikonal equations

$$p(x, \partial_x\phi_+(x,z)) = z, \quad \text{and} \quad \overline{p}(x, \partial_x\phi_-(x,z)) = \overline{z},$$

where p is as in (1.1.7). Furthermore, e_{wkb} and f_{wkb} are exponentially precise quasimodes since we have that

$$\|(P_h - z)e_{wkb}\|_2 = \mathscr{O}\left(\sqrt{h}\mathrm{e}^{-\frac{S}{h}}\right), \quad \text{and} \quad \left\|(P_h - z)^* f_{wkb}\right\|_2 = \mathscr{O}\left(\sqrt{h}\mathrm{e}^{-\frac{S}{h}}\right),$$

where $S = S(z)$ is as in Definition 1.2.2. These estimates can be obtained similar to the proof of Proposition 2.1.7.

The factors $a(z;h)$ and $b(z;h)$ are the asymptotic expansions of the normalization coefficients and it is easy to see that for all $\beta \in \mathbb{N}^2$

$$\partial^\beta_{z\overline{z}} a(z;h), \partial^\beta_{z\overline{z}} b(z;h) = \mathscr{O}(h^{-|\beta|}). \tag{2.1.9}$$

We have the following explicit expressions for the leading terms of $a(z;h)$ and $b(z;h)$.

Lemma 2.1.3.

$$a_0 = \left(\frac{-\mathrm{Im}\, g'(x_+)}{\pi}\right)^{\frac{1}{4}}, \textit{ and } b_0 = \left(\frac{\mathrm{Im}\, g'(x_-)}{\pi}\right)^{\frac{1}{4}}. \tag{2.1.10}$$

Proof. We will show the proof only for a_0^i since the statement for b_0^i can be achieved by analogous steps. To gain the asymptotic expansion of the normalization coefficient use the stationary phase method to calculate

$$I_h := h^{-\frac{1}{2}} \int \chi_e(x,z;h)^2 e^{\frac{-\Phi(x,z)}{h}} dx,$$

where

$$-\Phi(x,z) := i\phi_+(x,z) - \overline{i\phi_+(x,z)} = -2\mathrm{Im} \int_{x_+(z)}^x (z - g(y))dy.$$

On the support of χ_e the phase $\Phi(x,z)$ has the unique critical point $x = x_+(z)$ and it is non-degenerate since $\partial^2_{xx}\Phi(x_+(z),z) = -2\mathrm{Im}\, g'(x_+(z)) > 0$. Thus the Morse Lemma (see e.g.: [29]) guarantees the existence of a local $\mathscr{C}^\infty$ diffeomorphism $\kappa : V \to U$, where $V \subset \mathbb{R}$ is a neighborhood of $x_+(z)$ and $U \subset \mathbb{R}$ is a neighborhood of 0, such that

$$\Phi(\kappa^{-1}(x), z) = \Phi(x_+(z), z) + \frac{x^2}{2},$$

$\kappa^{-1}(0) = x_+(z)$ and

$$\frac{d\kappa}{dx}(x_+(z)) = |\partial^2_{xx}\Phi(x_+(z),z)|^{\frac{1}{2}} = \sqrt{-2\mathrm{Im}\, g'(x_+(z))} \neq 0. \tag{2.1.11}$$

Let $\chi \in \mathscr{C}_0^\infty(\mathbb{R})$ be supported in a small enough neighborhood of $x_+(z)$, assume that $1 \geq \chi \geq 0$ and suppose that $\chi \equiv 1$ near $x_+(z)$. One then gets that

$$I_h = \sqrt{2\pi} \sum_{n=0}^{N} \frac{1}{n!} \left(\frac{h}{2}\right)^n (\Delta^n u)(0) + \mathscr{O}(h^{N+1})$$

with $u(y) = \chi_e(\kappa^{-1}(y), z)^2 \chi(\kappa^{-1}(y)) |\kappa'(\kappa^{-1}(y))|^{-1}$. Since $u(0) = (-2\mathrm{Im}\, g'(x_+(z)))^{-1/2}$,

$$I_h = \left(\frac{\pi}{-\mathrm{Im}\, g'(x_+(z))}\right)^{\frac{1}{2}} + \mathscr{O}(h). \qquad \square$$

By the natural projection $\Pi : \mathbb{R} \to S^1$ as in Section 1.1.1 we can identify

$$\mathscr{C}_0^\infty(]x_+, x_+ + 2\pi[) = \{u \in \mathscr{C}^\infty(S^1) : \ x_+ \notin \mathrm{supp}\, u\}$$

and

$$\mathscr{C}_0^\infty(]x_- - 2\pi, x_-[) = \{u \in \mathscr{C}^\infty(S^1) : \ x_- \notin \mathrm{supp}\, u\},$$

with the slight abuse of notation that on the left hand side $x_\pm \in \mathbb{R}$ and on the right hand side $x_\pm \in S^1$. This identification permits us to define $e_{wkb}(x,z;h), f_{wkb}(x,z;h)$ on $\mathscr{C}^\infty(S^1)$.

2.1.2 – Quasimodes close to the boundary of Σ

Now let $z \in \Omega_\eta^a$. Following [4], we will construct quasimodes for the operator $P_h - z$, for z close to the boundary of Σ, by looking at the rescaled operator $\widetilde{P}_{\widetilde{h}} - \widetilde{z}$ as defined in (2.1.3).

Let us first note that $\frac{i}{h}\phi_+(x,z)$ and $\frac{i}{h}\phi_-(x,z)$ have the following behavior under the rescaling described at the beginning of this section:

$$\frac{i}{h}\phi_+(x,z) = \frac{i}{h}\int_{x_+}^{x} \big(z - g(y)\big)\, dy = \frac{i}{\widetilde{h}}\int_{\widetilde{x}_+}^{\widetilde{x}} \big(\widetilde{z} - \widetilde{g}(\widetilde{y})\big)\, d\widetilde{y} =: \frac{i}{\widetilde{h}}\widetilde{\phi}_+(\widetilde{x},\widetilde{z}) \tag{2.1.12}$$

and analogously for $\frac{i}{h}\phi_-(x,z)$. Taylor expansion shows us that the rescaled phases $\widetilde{\phi}_\pm(\widetilde{x},\widetilde{z})$ have for $z \in \Omega_\eta^a$ a non-degenerate critical point $\widetilde{x}_\pm(\widetilde{z})$ and they satisfy the relation

$$x_\pm(z) = \sqrt{\eta}\,\widetilde{x}_\pm(\widetilde{z}). \tag{2.1.13}$$

It is easy to see that locally

$$(\widetilde{P}_{\widetilde{h}} - \widetilde{z}) \mathrm{e}^{\frac{i}{\widetilde{h}}\widetilde{\phi}_+(\widetilde{x},\widetilde{z})} = 0,$$

Thus, the natural choice of quasimodes for $z \in \Omega \cap \Omega_\eta^a$ in the rescaled variables is as follows.

Proposition 2.1.4. *Let $\Omega \Subset \Sigma$, $z \in \Omega \cap \Omega_\eta^a$ and set $\widetilde{h} := \frac{h}{\eta^{3/2}}$. Then there exist functions*

$$a^\eta(\widetilde{z};\widetilde{h}) \sim a_0^\eta(\widetilde{z}) + \widetilde{h} a_1^\eta(\widetilde{z}) + \cdots \neq 0, \quad b^\eta(\widetilde{z};\widetilde{h}) \sim b_0^\eta(\widetilde{z}) + \widetilde{h} b_1^\eta(\widetilde{z}) + \cdots \neq 0,$$

depending smoothly on $\widetilde{z}$ such that all $\widetilde{z}$- and $\overline{\widetilde{z}}$-derivatives remain bounded as $h \to 0$ and $h^{\frac{2}{3}} < \eta \to 0$, such that

$$e_{wkb}^\eta(\widetilde{x},\widetilde{z};\widetilde{h}) := \big(\widetilde{h}\eta\big)^{-\frac{1}{4}} a^\eta(\widetilde{z};\widetilde{h}) \chi_e(\widetilde{x},\widetilde{z};\widetilde{h}) e^{\frac{i}{\widetilde{h}}\widetilde{\phi}_+(\widetilde{x},\widetilde{z})} \text{ and}$$
$$f_{wkb}^\eta(\widetilde{x},\widetilde{z};\widetilde{h}) := \big(\widetilde{h}\eta\big)^{-\frac{1}{4}} b^\eta(\widetilde{z};\widetilde{h}) \chi_f(\widetilde{x},\widetilde{z};\widetilde{h}) e^{\frac{i}{\widetilde{h}}\widetilde{\phi}_-(\widetilde{x},\widetilde{z})},$$

are $L^2(S^1/\sqrt{\eta}, \sqrt{\eta}d\widetilde{x})$-normalized. Here, $\chi_{e,f}$ are as in Definition 2.1.2. Furthermore,

$$a_0^\eta(\widetilde{z}) = \left(\frac{|\mathrm{Im}\, g''(a)(\widetilde{x}_+(\widetilde{z}) - a/\sqrt{\eta})(1+o(1))|}{\pi}\right)^{\frac{1}{4}}, \quad z \in \Omega_\eta^a,$$

$$b_0^\eta(\widetilde{z}) = \left(\frac{|\mathrm{Im}\, g''(a)(\widetilde{x}_-(\widetilde{z}) - a/\sqrt{\eta})(1+o(1))|}{\pi}\right)^{\frac{1}{4}}, \quad z \in \Omega_\eta^a.$$

Remark 2.1.5. In Proposition 2.1.4, we stated the Taylor expansion of the first order terms of $a^\eta(z;h)$ and $b^\eta(z;h)$. However, note that we have

$$a_0(z) = \left(\frac{-\operatorname{Im} g'(x_+(z))}{\pi}\right)^{\frac{1}{4}} = \eta^{\frac{1}{8}} a_0^\eta(\widetilde{z}),$$

where a_0 is the first order term of the normalization coefficient a of the quasimode e_{wkb}; see Lemma 2.1.3. Similar for b_0^η.

Proof. We will consider the proof only for the case of e_{wkb}^η since the case of f_{wkb}^η is the same.

By (2.1.13), (2.1.6) one computes that

$$\chi_e(\sqrt{\eta}\widetilde{x}, z; h/\eta^{1/2}) = \chi_e(\widetilde{x}, \widetilde{z}; \widetilde{h})$$

Consider $\|\chi_e(\cdot, z; h/\eta^{1/2}) e^{\frac{i}{h}\phi_+(\cdot,z)}\|^2_{L^2(S^1)}$ and perform the change of variables $x = \sqrt{\eta}\widetilde{x}$. Hence,

$$\int \chi_e(x, z; h/\eta^{1/2})^2 e^{-\frac{2}{h}\operatorname{Im}\phi_+(x,z)} dx = \sqrt{\eta} \int \chi_e\left(\widetilde{x}, \widetilde{z}; \widetilde{h}\right)^2 e^{-\frac{2}{\widetilde{h}}\operatorname{Im}\int_{\widetilde{x}_+}^{\widetilde{x}}(\widetilde{z}-\widetilde{g}(\widetilde{y}))d\widetilde{y}} d\widetilde{x}. \tag{2.1.14}$$

The stationary phase method yields that (2.1.14)$\sim \sqrt{\eta}\widetilde{h}^{\frac{1}{2}}(\widetilde{c}_0(\widetilde{z}) + \widetilde{h}\widetilde{c}_1(\widetilde{z}) + \dots)$, where the $\widetilde{c}_j(\widetilde{z})$ depend smoothly on $\widetilde{z}$ such that all $\widetilde{z}$- and $\overline{\widetilde{z}}$-derivatives remain bounded as $h \to 0$ and $h^{\frac{2}{3}} < \eta \to 0$.

On the other hand, the stationary phase method applied to $\|\chi_e e^{\frac{i}{h}\phi_+}\|^2$ (compare with Section 2.1.1) yields that

$$\|\chi_e(\cdot, z; h) e^{\frac{i}{h}\phi_+(\cdot,z)}\|^2_{L^2(S^1)} \sim h^{\frac{1}{2}}(c_0(z) + hc_1(z) + \dots)$$

with

$$c_0(z) = \left(\frac{\pi}{-\operatorname{Im} g'(x_+(z))}\right)^{\frac{1}{2}}.$$

Since $\chi_e(x, z; h) \equiv \chi_e(x, z; h/\eta^{1/2})$ locally around $x_+(z)$, we may conclude that for all $k \in \mathbb{N}_0$

$$\widetilde{c}_k(\widetilde{z}) = \eta^{\frac{3k}{2}+\frac{1}{4}} c_j(z).$$

In particular, the Taylor expansion around the critical point a yields that

$$\widetilde{c}_0(\widetilde{z}) = \left(\frac{\pi}{|\operatorname{Im} g''(a)(\widetilde{x}_+(\widetilde{z}) - a/\sqrt{\eta})(1 + o(1))|}\right)^{\frac{1}{2}}, \quad z \in \Omega_\eta^a.$$

Thus, we conclude the statement of the proposition. □

Considering the above describe quasimodes in the original variable $x \in S^1$ leads to the following

Definition 2.1.6. Let $\Omega \Subset \Sigma$, $z \in \Omega \cap \Omega_\eta^a$ and set $\widetilde{h} := \frac{h}{\eta^{3/2}}$. Then define

$$e_{wkb}^\eta(x, z; h) := \left(\frac{h}{\eta^{1/2}}\right)^{-\frac{1}{4}} a^\eta(\widetilde{z}; \widetilde{h}) \chi_e^\eta(x, z; h/\eta^{1/2}) e^{\frac{i}{h}\phi_+(x,z)} \text{ and}$$

$$f_{wkb}^\eta(x, z; h) := \left(\frac{h}{\eta^{1/2}}\right)^{-\frac{1}{4}} b^\eta(\widetilde{z}; \widetilde{h}) \chi_f^\eta(x, z; h/\eta^{1/2}) e^{\frac{i}{h}\phi_-(x,z)},$$

where $\chi_{e,f}^\eta(x, z; h/\eta^{1/2}) = \chi_{e,f}(x, z; h/\eta^{1/2})$. We choose this notation to make the distinctions between the two cases $z \in \Omega_i$ and $z \in \Omega_\eta^a$ more apparent.

Furthermore, we have the following estimates for the precision of the quasimodes e_{wkb}^η and f_{wkb}^η:

$$\left\|(P_h - z)e_{wkb}^\eta\right\|_2 = \mathcal{O}\left(h^{1/2}\eta^{1/4} \mathrm{e}^{-\frac{S}{h}}\right), \quad \text{and} \quad \left\|(P_h - z)^* f_{wkb}^\eta\right\|_2 = \mathcal{O}\left(h^{1/2}\eta^{1/4} \mathrm{e}^{-\frac{S}{h}}\right),$$

where $S = S(z)$ is as in Definition 1.2.2 (recall as well that $S \asymp \eta^{3/2}$, cf. (2.1.5)). These estimates can be obtained similar to the proof of Proposition 2.1.7.

2.1.3 – Approximation of the eigenfunctions of $Q(z)$ and $\widetilde{Q}(z)$

Recall Q and $\widetilde{Q}$ given in Section 1.2.2. We will use the above defined quasimodes to prove estimates on the lowest eigenvalue of Q, t_0^2. Furthermore, we will give estimates on the approximation of the eigenfunctions e_0 and f_0 by the quasimodes e_{wkb} and f_{wkb}. We will prove an extended version of a result in [67, Sec. 7.2 and 7.4].

Proposition 2.1.7. *Let $z \in \Omega \Subset \Sigma$ and let $S = S(z)$ be defined as in Definition 1.2.2. Then, for $h^{\frac{2}{3}} \ll \eta \leq 1/C$*

$$t_0^2(z) \leq \mathcal{O}\left(\eta^{\frac{1}{2}} h e^{-\frac{2S}{h}}\right).$$

Furthermore, there exists a constant $C > 0$, uniform in $z \in \Omega$, such that

$$t_1^2(z) - t_0^2(z) \geq \frac{\eta^{\frac{1}{2}} h}{C}$$

for $h > 0$ small enough.

Remark 2.1.8. The case $z \in \Omega$ with $\mathrm{dist}(\Omega, \partial\Sigma) > 1/C$ has been proven in [67, Sec. 7.1]. Since it will be useful further on we shall give a proof of the statement and indicate how to deduce the statement in the case of $z \in \Omega \cap \Omega_\eta^a$.

Proof. Let us first suppose that $z \in \Omega_i$ (cf. Section 2.1). Recall the definition of the self-adjoint operator $Q(z)$ given in (1.2.4) and define

$$r := r(x,z;h) := Q(z) e_{wkb}(x,z;h). \tag{2.1.15}$$

Recall, by (2.1.7), that $e_{wkb}(x,z;h) = h^{-\frac{1}{4}} a(z;h) \chi_e(x,z;h) e^{\frac{i}{h}\phi_+(x,z)}$. Since $x_-(z)$ is smooth in z and all its z- and $\overline{z}$-derivatives are independent of h, it follows from (2.1.6) that for all $\alpha \in \mathbb{N}^3 \backslash \{0\}$

$$\partial^\alpha_{z\overline{z}x} \chi_e(x,z;h) = \mathcal{O}\left(h^{-\frac{|\alpha|}{2}}\right), \tag{2.1.16}$$

with support in $X_- :=]x_- - 2\pi, x_- - 2\pi + h^{1/2}[\cup]x_- - h^{1/2}, x_-[$. By definition of $\phi_+(x,z)$

$$(P_h - z) e^{\frac{i}{h}\phi_+(x,z)} = 0$$

for $x \in]x_- - 2\pi, x_-[$. This implies

$$\begin{aligned} (P_h - z) e_{wkb}(x,z) &= h^{-\frac{1}{4}} a(z;h) [(P_h - z), \chi_e(x,z;h)] e^{\frac{i}{h}\phi_+(x,z)} \\ &= h^{-\frac{1}{4}} a(z;h) \frac{h}{i} \partial_x \chi_e(x,z;h) e^{\frac{i}{h}\phi_+(x,z)}. \end{aligned} \tag{2.1.17}$$

Continuing, one computes that

$$\begin{aligned} &(P_h - z)^*(P_h - z) e_{wkb}(x,z) = \\ &a(z;h) \frac{h^{\frac{3}{4}}}{i} \left\{ \frac{h}{i} \partial^2_{xx} \chi_e(x,z;h) + \partial_x \chi_e(x,z;h) \left(\partial_x \phi_+ + \overline{g(x) - z} \right) \right\} e^{\frac{i}{h}\phi_+}. \end{aligned} \tag{2.1.18}$$

where $\phi_+ = \phi_+(x,z)$. Since for $x \in X_-$

$$\partial_x \phi_+(x,z) + \overline{g(x) - z} = z - g(x) + \overline{g(x) - z} = -2i \mathrm{Im}\left(g(x) - z\right) = \mathcal{O}\left(h^{\frac{1}{2}}\right), \tag{2.1.19}$$

it follows from (2.1.16), (2.1.18) that

$$r = Q(z) e_{wkb}(x,z) = \mathcal{O}\left(h^{\frac{3}{4}}\right) e^{\frac{i}{h}\phi_+(x,z)}, \tag{2.1.20}$$

which has its support in X_-. Thus,

$$(e_{wkb} | Q(z) e_{wkb}) = \int \mathcal{O}\left(h^{\frac{1}{2}}\right) \mathbb{1}_{X_-}(x) e^{-\frac{\Phi(x,z)}{h}} dx, \tag{2.1.21}$$

where $\Phi(x,z) = 2\int_{x_+(z)}^{x} \mathrm{Im}\,(z-g(y))dy$. By Taylor's formula

$$\begin{cases} \Phi(x,z) = \Phi(x_-(z),z) + \mathcal{O}(h), \text{ for } x \in]x_- - h^{1/2}, x_-[\\ \Phi(x,z) = \Phi(x_-(z)-2\pi,z) + \mathcal{O}(h), \text{ for } x \in]x_- - 2\pi, x_- - 2\pi + h^{1/2}[\end{cases}$$

and thus

$$\mathrm{e}^{-\frac{\Phi(x,z)}{h}} \leq \mathcal{O}\left(\mathrm{e}^{-\frac{2S}{h}}\right),$$

where $S = \min\left(\mathrm{Im}\int_{x_+}^{x_-}(z-g(y))dy, \mathrm{Im}\int_{x_+}^{x_--2\pi}(z-g(y))dy\right)$. Hence,

$$(e_{wkb}|Q(z)e_{wkb}) \leq \mathcal{O}\left(h^{\frac{1}{2}}\mathrm{e}^{-\frac{2S}{h}}\right)\int 1\!\!1_{X_-}(x)dx = \mathcal{O}\left(h\mathrm{e}^{-\frac{2S}{h}}\right), \tag{2.1.22}$$

and, since Q is self-adjoint, it follows that $t_0^2(z) = \mathcal{O}\left(he^{-\frac{2S(z)}{h}}\right)$. Similarly, one computes that

$$\|r\|^2 = \mathcal{O}\left(h^2\mathrm{e}^{-\frac{2S}{h}}\right). \tag{2.1.23}$$

The proof of the desired statement about $t_1^2(z) - t_0^2(z)$ for $z \in \Omega_i$ can be found in the proof of Proposition 7.2 in [67, Sec. 7.1].

Suppose now that $z \in \Omega \cap \Omega_\eta^a$. The desired statement follows by a rescaling argument. Recall (2.1.3) and, using the quasimodes $e_{wkb}^{\eta}(x,z)$, note that

$$t_0^2(Q(z)) = t_0^2(\eta^2(\widetilde{P}_{\widetilde{h}} - \widetilde{z})^*(\widetilde{P}_{\widetilde{h}} - \widetilde{z})) = \mathcal{O}\left(\eta^2\widetilde{h}e^{-\frac{2\widetilde{S}(\widetilde{z})}{\widetilde{h}}}\right),$$

where $\widetilde{S}$ is defined in the obvious way via $\widetilde{\phi}_+$ and

$$\frac{\widetilde{S}(\widetilde{z})}{\widetilde{h}} = \frac{S(z)}{h}. \tag{2.1.24}$$

Hence,

$$t_0^2(z) = \mathcal{O}\left(h\eta^{1/2}e^{-\frac{2S(z)}{h}}\right). \tag{2.1.25}$$

The estimate on $t_1^2(z) - t_0^2(z)$ in the case $z \in \Omega \cap \Omega_\eta^a$ can be deduced as well by a rescaling argument: note that $t_1^2(Q(z)) = t_1^2(\eta^2(\widetilde{P}_{\widetilde{h}} - \widetilde{z})^*(\widetilde{P}_{\widetilde{h}} - \widetilde{z}))$. The statement then follows by performing the same steps of the proof of Proposition 7.2 in [67, Sec. 7.1] in the rescaled space $L^2(S^1/\sqrt{(\eta)}, \sqrt{\eta}d\widetilde{x})$ and using the quasimode $e_{wkb}^{\eta}(x,z)$ together with the estimate given in Proposition 4.3.5 in [4]. □

Proposition 2.1.9. *Let $z \in \Omega \Subset \Sigma$. Then the eigenvalue $t_0^2(z)$ is a smooth function of z and the eigenfunctions $e_0(z)$ and $f_0(z)$ can be chosen to have the same property.*

Proof. Let us suppose first that $z \in \Omega_i$. The operator $Q(z)$ is bounded in $H^2(S^1) \to L^2(S^1)$ and in norm real-analytic in z since for $z_0 \in \Omega$

$$Q(z) = Q(z_0) - (P - z_0)^*(z - z_0) - (P - z_0)(\overline{z - z_0}) + |z - z_0|^2. \tag{2.1.26}$$

Let ζ be in the resolvent set $\rho(Q(z))$ of $Q(z)$ and consider the resolvent

$$R(\zeta, Q(z)) := (\zeta - Q(z))^{-1}.$$

By [41, II - §1.3] we know that the resolvent depends locally analytically on the variables ζ and z. More precisely if $\zeta_0 \notin \sigma(Q(z_0))$ for $z_0 \in \Omega$ then $R(\zeta, Q(z))$ is holomorphic in ζ and real-analytic in z in a small neighborhood of ζ_0 and in a small neighborhood of z_0.

Remark 2.1.10. The proof in [41, II - §1.3] is given in the case of finite dimensional spaces. However, it can be extended directly to bounded operators on Banach spaces.

By [41, IV - §3.5] we know that the simple eigenvalue $t_0^2(z)$ depends continuously on $Q(z)$. Thus, by Proposition 2.1.7 and the continuity of $t_0^2(z)$ there exists, for $h > 0$ small enough, a constant $D > 0$ such that for all z in a neighborhood of a point $z_0 \in \Omega$

$$t_1^2(z) > \frac{h}{D}.$$

Define γ to be the positively oriented circle of radius $h/(2D)$ centered at 0 and consider the spectral projection of $Q(z)$ onto the eigenspace associated with $t_0^2(z)$

$$\Pi_{t_0^2}(z) = \frac{1}{2\pi i}\int_\gamma R(\zeta, Q(z))d\zeta.$$

Since the resolvent $R(\zeta, Q(z))$ is smooth in z it follows that $\Pi_{t_0^2}(z)$ is smooth in z. Now set $e(x,z)$ to be a smooth quasimode for $P_h - z$ for $z \in \Omega_i$ as in Section 2.1 which depends smoothly on z. Thus, by setting

$$e_0(x,z,h) = \frac{\Pi_{t_0^2}(z)e_{wkb}(x,z,h)}{\|\Pi_{t_0^2}(z)e_{wkb}(-,z,h)\|},$$

we deduce that also $e_0(x,z)$ depends smoothly on z. The statement for $f_0(z)$ follows by performing the same argument for $\widetilde{Q}(z)$ instead of $Q(z)$ and with the quasimode f_{wkb}.

Using that $\Pi_{t_0^2}(z)$ and $Q(z)$ are smooth and that the operator $\Pi_{t_0^2}Q\Pi_{t_0^2}$ has finite rank we see by

$$t_0^2(z) = \operatorname{tr}\left(\Pi_{t_0^2}(z)Q(z)\Pi_{t_0^2}(z)\right)$$

that $t_0^2(z)$ is smooth.

In the case of $z \in \Omega \cap \Omega_\eta^a$ for $h^{2/3} \ll \eta < \text{const.}$ we follow the exact same steps as above, mutatis mutandis. We take the estimate $t_1^2(z) > \frac{h\sqrt{\eta}}{D}$ for z in a neighborhood of a fixed $z_0 \in \Omega \cap \Omega_\eta^a$ (following from Proposition 2.1.7) and thus we pick, as above, $\widetilde{\gamma}$ to be the positively oriented circle of radius $h\sqrt{\eta}/(2D)$ centered at 0. Hence, for $z \in \Omega \cap \Omega_\eta^{a,b}$

$$\Pi_{t_0^2}(z) = \frac{1}{2\pi i}\int_{\widetilde{\gamma}} R(\zeta, Q(z))d\zeta, \quad e_0(x,z,h) = \frac{\Pi_{t_0^2}(z)e^\eta_{wkb}(x,z,h)}{\|\Pi_{t_0^2}(z)e^\eta_{wkb}(-,z,h)\|}.$$

Following the same arguments as above we conclude the statement of the proposition also in the case of $z \in \Omega \cap \Omega_\eta^a$. $\square$

Proposition 2.1.11. *Let $z \in \Omega \Subset \Sigma$ and let e_0 and f_0 be the eigenfunctions of the operators Q and $\widetilde{Q}$ with respect to their smallest eigenvalue (as in Section 2.2.1). Let $S = S(z)$ be defined as in Definition 1.2.2. Then*

- *for $z \in \Omega$ with $\operatorname{dist}(\Omega, \partial\Sigma) > 1/C$ and for all $\beta \in \mathbb{N}^2$*

$$\|\partial^\beta_{z\bar{z}}(e_0 - e_{wkb})\|,\ \|\partial^\beta_{z\bar{z}}(f_0 - f_{wkb})\| = \mathcal{O}\left(h^{-|\beta|}e^{-\frac{S}{h}}\right). \tag{2.1.27}$$

  Furthermore, the various z- and $\bar{z}$-derivatives of e_0, f_0, e_{wkb} and f_{wkb} have at most temperate growth in $1/h$, more precisely for all $\beta \in \mathbb{N}^2$

$$\|\partial^\beta_{z\bar{z}}e_{wkb}\|,\ \|\partial^\beta_{z\bar{z}}f_{wkb}\|,\ \|\partial^\beta_{z\bar{z}}e_0\|,\ \|\partial^\beta_{z\bar{z}}f_0\| = \mathcal{O}\left(h^{-|\beta|}\right); \tag{2.1.28}$$

- *for $h^{2/3} \ll \eta < const.$, $z \in \Omega \cap \Omega_\eta^a$ and for all $\beta \in \mathbb{N}^2$*

$$\|\partial^\beta_{z\bar{z}}(e_0 - e^\eta_{wkb})\|,\ \|\partial^\beta_{z\bar{z}}(f_0 - f^\eta_{wkb})\| = \mathcal{O}\left(\eta^{\frac{|\beta|}{2}}h^{-|\beta|}e^{-\frac{S}{h}}\right). \tag{2.1.29}$$

Furthermore, the various z- and $\bar{z}$-derivatives of e_0, f_0, e^{η}_{wkb} and f^{η}_{wkb} have at most temperate growth in $\sqrt{\eta}/h$, more precisely

$$\|\partial^{\beta}_{z\bar{z}}e^{\eta}_{wkb}\|,\ \|\partial^{\beta}_{z\bar{z}}f^{\eta}_{wkb}\|,\ \|\partial^{\beta}_{z\bar{z}}e_0\|,\ \|\partial^{\beta}_{z\bar{z}}f_0\| = \mathcal{O}\left(\eta^{\frac{|\beta|}{2}}h^{-|\beta|}\right) \tag{2.1.30}$$

for all $\beta \in \mathbb{N}^2$.

Remark 2.1.12. Let us recall that

- for $z \in \Omega \Subset \mathring{\Sigma}$, in the case where Ω is independent of $h>0$ and has a positive distance to the boundary of Σ we have $1/C \leq S \leq C$ for some constant $C>0$. Thus, we may formulate the corresponding estimates of Proposition 2.1.11 uniformly in z;
- for $h^{2/3} \ll \eta < const.$ and $z \in \Omega \cap \Omega^{a}_{\eta}$ (2.1.5) implies estimates uniform in z but η dependent.

This implies the following

Corollary 2.1.13. *Under the assumptions of Proposition 2.1.11,*

- *for $z \in \Omega_i$ there exists a constant $C>0$ such that for all $\beta \in \mathbb{N}^2$*

$$\|\partial^{\beta}_{z\bar{z}}(e_0 - e_{wkb})\|,\ \|\partial^{\beta}_{z\bar{z}}(f_0 - f_{wkb})\| = \mathcal{O}\left(h^{-|\beta|}\mathrm{e}^{-\frac{1}{Ch}}\right); \tag{2.1.31}$$

- *for $h^{2/3} \ll \eta < const.$, $z \in \Omega \cap \Omega^{a,b}_{\eta}$ and for all $\beta \in \mathbb{N}^2$*

$$\|\partial^{\beta}_{z\bar{z}}(e_0 - e^{\eta}_{wkb})\|,\ \|\partial^{\beta}_{z\bar{z}}(f_0 - f^{\eta}_{wkb})\| = \mathcal{O}\left(\eta^{\frac{|\beta|}{2}}h^{-|\beta|}\mathrm{e}^{-\frac{\asymp\eta^{3/2}}{h}}\right). \tag{2.1.32}$$

Remark 2.1.14. The proof of Proposition 2.1.11 is unfortunately somewhat long and technical and we have split it into several lemmas. Furthermore, we will only be discussing the results for $e_{wkb}(z)$, $e^{\eta}_{wkb}(z)$ and $e_0(z)$, since the others can be obtained similarly.

Lemma 2.1.15. *Let $\Omega \Subset \Sigma$ such that* $\operatorname{dist}(\Omega, \partial\Sigma) > 1/C$. *For $z \in \Omega$ define $r := r(x,z;h) := Q(z)e_{wkb}(x,z)$ as in (2.1.15). Then, for all $\beta \in \mathbb{N}^2$,* $\operatorname{supp}\partial^{\beta}_{z\bar{z}}r \subset]x_- - 2\pi, x_- - 2\pi + h^{1/2}[\cup]x_- - h^{1/2}, x_-[$ *and*

$$\|\partial^{\beta}_{z\bar{z}}r\| = \mathcal{O}\left(h^{1-|\beta|}\mathrm{e}^{-\frac{S}{h}}\right).$$

Proof. Using (2.1.16), (2.1.18) we conclude by the Leibniz rule that for $\beta \in \mathbb{N}^2$

$$\partial^{\beta}_{z\bar{z}}r = \mathcal{O}\left(h^{\frac{3}{4}-|\beta|}\right)e^{\frac{i}{h}\phi_+(x,z)}$$

which is supported in $]x_- - 2\pi, x_- - 2\pi + h^{1/2}[\cup]x_- - h^{1/2}, x_-[$ and one computes that $\|\partial^{\beta}_{z\bar{z}}r\|^2 = \mathcal{O}\left(h^{2-2|\beta|}\mathrm{e}^{-\frac{2S}{h}}\right)$. ∎

Lemma 2.1.16. *Let $\Omega \Subset \Sigma$ such that* $\operatorname{dist}(\Omega, \partial\Sigma) > 1/C$ *and let $z \in \Omega$. Moreover, let $\Pi_{t_0^2} : L^2(S^1) \to \mathbb{C}e_0$ denote the spectral projection of $Q(z)$ onto the eigenspace associated with t_0^2. Then,*

$$\|\partial^{\beta}_{z\bar{z}}\Pi_{t_0^2}(z)\|_{L^2\to H^2_{sc}} = \mathcal{O}\left(h^{-\frac{|\beta|}{2}}\right).$$

Proof. By virtue of Proposition 2.1.7 and the continuity of $t_0^2(z)$ there exists for $h>0$ small enough a constant $D>0$ such that for all z in a neighborhood of a point $z_0 \in \Omega$

$$t_1^2(z) > \frac{h}{D}.$$

Let γ be the positively oriented circle of radius $h/(2D)$ centered at 0. Note that γ is locally independent of z. Thus, we gain a path such that $0, t_1^2(z) \notin \gamma$ and which has length $|\gamma| = h\pi/D$. For $\lambda \in \gamma$ we have that

$$\|(\lambda - Q(z))^{-1}\| = \frac{1}{\operatorname{dist}(\lambda, \sigma(Q(z)))} = \mathcal{O}(|\gamma|^{-1}). \tag{2.1.33}$$

By (2.1.26) and the resolvent identity we see that

$$\partial_z(\lambda - Q(z))^{-1} = -(\lambda - Q(z))^{-1}(P_h - z)^*(\lambda - Q(z))^{-1} \tag{2.1.34}$$

as well as

$$\partial_{\bar{z}}(\lambda - Q(z))^{-1} = -(\lambda - Q(z))^{-1}(P_h - z)(\lambda - Q(z))^{-1}. \tag{2.1.35}$$

Similarly, we see that the higher derivatives $\partial_z^n \partial_{\bar{z}}^m (\lambda - Q(z))^{-1}$, for $(n,m) \in \mathbb{N}^2 \backslash \{0\}$, are finite linear combinations of terms of the form

$$(\lambda - Q(z))^{-1}\partial_{z\bar{z}}^{\alpha_1}(Q(z))(\lambda - Q(z))^{-1} \cdots \partial_{z\bar{z}}^{\alpha_k}(Q(z))(\lambda - Q(z))^{-1} \tag{2.1.36}$$

with $\alpha_j = (1,0),(0,1),(1,1)$ and $\alpha_1 + \cdots + \alpha_k = (n,m)$. Thus it is sufficient to estimate the terms of the form $(P_h - z)(Q(z) - \lambda)^{-1}$ and $(P_h - z)^*(Q(z) - \lambda)^{-1}$. Since $Q(z) = (P_h - z)^*(P_h - z)$, it follows that

$$\|(P_h - z)u\|^2 - |\gamma|\|u\|^2 \leq |((Q(z) - \lambda)u|u)| \leq \|(Q(z) - \lambda)u\|\|u\|. \tag{2.1.37}$$

Since $Q(z) > 0$ is self-adjoint and since $\operatorname{dist}(\lambda, \sigma(Q(z))) \asymp |\gamma|$ we have the a priori estimate

$$\|(Q(z) - \lambda)u\| \geq C|\gamma|\|u\|$$

for all $u \in H_{sc}^2(S^1)$, where $C > 0$ is a constant locally uniform in z. This implies

$$\begin{aligned}\|(P_h - z)u\|^2 &\leq \left(\|(Q(z) - \lambda)u\| + |\gamma|\|u\|\right)\|u\| \\ &\leq \widetilde{C}\|(Q(z) - \lambda)u\|\|u\| \leq \frac{C}{|\gamma|}\|(Q(z) - \lambda)u\|^2,\end{aligned}$$

where $C > 0$ is a constant uniform in z. Hence

$$\|(P_h - z)(Q(z) - \lambda)^{-1}\|_{L^2 \to L^2} = \mathcal{O}\left(|\gamma|^{-\frac{1}{2}}\right).$$

Finally, note that since $[P_h^*, P_h] = \mathcal{O}_{H_{sc}^2 \to L^2}(h)$ we can replace P_h by it's adjoint in (2.1.37) and gain the estimate

$$\|(P_h - z)^*(Q(z) - \lambda)^{-1}\|_{L^2 \to L^2} = \mathcal{O}\left(|\gamma|^{-\frac{1}{2}}\right).$$

Using (2.1.36) and the fact that $|\gamma| = h\pi/D$ we have that for all $\beta \in \mathbb{N}^2 \backslash \{0\}$

$$\|\partial_{z\bar{z}}^{\beta}(\lambda - Q(z))^{-1}\|_{L^2 \to H_{sc}^2} = \mathcal{O}\left(h^{-\frac{|\beta|+2}{2}}\right). \tag{2.1.38}$$

Since for $u \in L^2(S^1)$

$$\frac{1}{2\pi i}\int_{\gamma}(\lambda - Q(z))^{-1}u\,d\lambda = \Pi_{t_0^2}u,$$

(2.1.38) implies

$$\|\partial_{z\bar{z}}^{\beta}\Pi_{t_0^2}(z)\|_{L^2 \to H_{sc}^2} = \mathcal{O}\left(h^{-\frac{|\beta|}{2}}\right).$$

□

Lemma 2.1.17. *Under the assumptions of Lemma 2.1.16 we have*

$$\|\partial_{z\bar{z}}^{\beta}e_{wkb}(\cdot, z)\|, \|\partial_{z\bar{z}}^{\beta}\Pi_{t_0^2}e_{wkb}(\cdot, z)\| = \mathcal{O}\left(h^{-|\beta|}\right).$$

Proof. Using (2.1.7), one computes that

$$\partial_z e_{wkb}(x,z) = h^{-\frac{1}{4}}\Big\{ \partial_z \chi_e(x,z;h) a^i(z;h) + \chi_e(x,z;h)\partial_z a^i(z;h) + \chi_e(x,z;h) a^i(z;h)\frac{i}{h}\partial_z\phi_+(x,z)\Big\} \mathrm{e}^{\frac{i}{h}\phi_+(x,z)}.$$

By the triangular inequality, we get

$$\begin{aligned}\|\partial_z e_{wkb}(\cdot,z)\| \leq\, & h^{-\frac{1}{4}}\|\partial_z\chi_e(\cdot,z)a^i(z;h)\mathrm{e}^{\frac{i}{h}\phi_+(\cdot,z)}\| \\ &+ h^{-\frac{1}{4}}\|\chi_e(\cdot,z)\partial_z a^i(z;h)\mathrm{e}^{\frac{i}{h}\phi_+(\cdot,z)}\| \\ &+ h^{-\frac{1}{4}}\|\chi_e(\cdot,z)a^i(z;h)ih^{-1}\partial_z\phi_+(\cdot,z)\mathrm{e}^{\frac{i}{h}\phi_+(\cdot,z)}\|.\end{aligned}$$

Recalling from (2.1.16) that $\partial_z\chi_e(x,z;h) = \mathcal{O}(h^{-1/2})$ is supported in $]x_- - 2\pi, x_- - 2\pi + h^{1/2}[\cup]x_- - h^{1/2}, x_-[$, one computes

$$h^{-\frac{1}{4}}\|\partial_z\chi_e(\cdot,z)a^i(z;h)\mathrm{e}^{\frac{i}{h}\phi_+(\cdot,z)}\| = \mathcal{O}\left(h^{-\frac{1}{2}}\mathrm{e}^{-\frac{S}{h}}\right).$$

Using (2.1.9), the stationary phase method implies

$$h^{-\frac{1}{4}}\|\chi_e(\cdot,z)\partial_z a^i(z;h)\mathrm{e}^{\frac{i}{h}\phi_+(\cdot,z)}\| = \mathcal{O}(h^{-1}).$$

Furthermore, since

$$\partial_z\phi_+(x,z) = \int_{x_+(z)}^{x} dy - \xi_+(z)\partial_z x_+(z) \tag{2.1.39}$$

it follows by the stationary phase method that

$$h^{-\frac{1}{4}}\|\chi_e(\cdot,z)a^i(z;h)\frac{i}{h}\partial_z\phi_+(\cdot,z)\mathrm{e}^{\frac{i}{h}\phi_+(\cdot,z)}\| = \frac{1}{h}|\xi_+(z)\partial_z x_+(z)| + \mathcal{O}(1).$$

Hence, by putting all of the above together

$$\|\partial_z e_{wkb}(\cdot,z)\| = \mathcal{O}\left(h^{-1}\right).$$

Similarly, using (2.1.9), (2.1.16), the stationary phase method implies

$$\|\partial^\beta_{z\bar{z}} e_{wkb}(\cdot,z)\| = \mathcal{O}\left(h^{-|\beta|}\right).$$

Lemma 2.1.16 then implies by the Leibniz rule that

$$\|\partial^\beta_{z\bar{z}}\Pi_{t_0^2} e_{wkb}\| = \mathcal{O}\left(h^{-|\beta|}\right).$$

□

Remark 2.1.18. As in Lemma 2.1.17, we have for $z \in \Omega$ with $\mathrm{dist}(\Omega, \partial\Sigma) > 1/C$

$$\|\partial^\beta_{z\bar{z}} f_{wkb}(\cdot,z)\| = \mathcal{O}\left(h^{-|\beta|}\right)$$

and

$$\|\partial^\beta_{z\bar{z}}\widetilde{\Pi}_{t_0^2} f_{wkb}\| = \mathcal{O}\left(h^{-|\beta|}\right).$$

where $\widetilde{\Pi}_{t_0^2} : L^2(S^1) \to \mathbb{C} f_0$ is the spectral projection of $\widetilde{Q}(z)$ onto the eigenspace associated with the eigenvalue t_0^2.

Proof of Proposition 2.1.11. Part I - First, suppose that $z \in \Omega_i$. Let r be as in Lemma 2.1.15 and consider for $\lambda \in \mathbb{C}$

$$(\lambda - Q(z))e_{wkb} = \lambda e_{wkb} - r.$$

If $\lambda \notin \sigma(Q(z)) \cup \{0\}$ we have

$$(\lambda - Q(z))^{-1} e_{wkb} = \frac{1}{\lambda} e_{wkb} + \frac{1}{\lambda}(\lambda - Q(z))^{-1} r.$$

As in the proof of Lemma 2.1.15, define γ to be the positively oriented circle of radius $h/(2D)$ centered at 0. γ is locally independent of z. Thus, we gain a path such that $0, t_1^2(z) \notin \gamma$ and which has length $|\gamma| = h\pi/D$. Hence

$$\frac{1}{2\pi i}\int_\gamma (\lambda - Q(z))^{-1} e_{wkb} d\lambda = e_{wkb} + \frac{1}{2\pi i}\int_\gamma \frac{1}{\lambda}(\lambda - Q(z))^{-1} r d\lambda. \tag{2.1.40}$$

By Lemma 2.1.15, (2.1.25) and (2.1.33)

$$\left\| \frac{1}{2\pi i}\int_\gamma \frac{1}{\lambda}(\lambda - Q(z))^{-1} r d\lambda \right\| = \mathcal{O}\left(e^{-\frac{S}{h}}\right)$$

By (2.1.40)

$$\|\Pi_{t_0^2} e_{wkb} - e_{wkb}\| = \mathcal{O}\left(e^{-\frac{S}{h}}\right). \tag{2.1.41}$$

Recall that e_{wkb} is normalized. Pythagoras' theorem then implies

$$\|\Pi_{t_0^2} e_{wkb}\|^2 = \|e_{wkb}\|^2 - \|e_{wkb} - \Pi_{t_0^2} e_{wkb}\|^2 = 1 - \mathcal{O}\left(\mathrm{e}^{-\frac{2S}{h}}\right) \tag{2.1.42}$$

which yields

$$e_0 = \frac{1}{\|\Pi_{t_0^2} e_{wkb}\|}\Pi_{t_0^2} e_{wkb} = \left(1 + \mathcal{O}\left(e^{-\frac{2S}{h}}\right)\right)\Pi_{t_0^2} e_{wkb}. \tag{2.1.43}$$

Let us now turn to the z- and $\overline{z}$-derivatives of $e_0 - e_{wkb}$. By (2.1.43)

$$\begin{aligned}\left\|\partial_{z\overline{z}}^\beta (e_0(z) - e_{wkb}(z))\right\| &= \left\|\partial_{z\overline{z}}^\beta \left(\frac{\Pi_{t_0^2} e_{wkb}(z)}{\|\Pi_{t_0^2} e_{wkb}(z)\|} - e_{wkb}(z)\right)\right\| \\ &= \left\|\partial_{z\overline{z}}^\beta \left(\frac{(\Pi_{t_0^2} - 1)e_{wkb} + (1 - \|\Pi_{t_0^2} e_{wkb}\|)e_{wkb}}{\|\Pi_{t_0^2} e_{wkb}(z)\|}\right)\right\|.\end{aligned}$$

First, note that Lemma 2.1.28 together with (2.1.42) implies

$$\partial_{z\overline{z}}^\beta \|\Pi_{t_0^2} e_{wkb}\| = \mathcal{O}\left(h^{-|\beta|}\right). \tag{2.1.44}$$

Using this result and (2.1.42) implies by the Leibniz rule applied to (2.1.43) that

$$\|\partial_{z\overline{z}}^\beta e_0\| = \mathcal{O}\left(h^{-|\beta|}\right).$$

Next, applying Lemma 2.1.15 and (2.1.38) to (2.1.40) yields

$$\left\|\partial_{z\overline{z}}^\beta (\Pi_{t_0^2} - 1)e_{wkb}\right\| = \left\|\partial_{z\overline{z}}^\beta \frac{1}{2\pi i}\int_\gamma \frac{1}{\lambda}(\lambda - Q(z))^{-1} r d\lambda\right\| = \mathcal{O}\left(h^{-|\beta|}\mathrm{e}^{-\frac{S}{h}}\right). \tag{2.1.45}$$

Using (2.1.42) and the fact that e_{wkb} is normalized, it follows that

$$1 - \|\Pi_{t_0^2} e_{wkb}\| = \frac{\left\|(\Pi_{t_0^2} - 1)e_{wkb}\right\|^2}{1 + \|\Pi_{t_0^2} e_{wkb}\|}.$$

This, together with (2.1.45), (2.1.44) and the Leibniz rule imply that

$$\partial^{\beta}_{z\bar{z}}(1 - \|\Pi_{t_0^2} e_{wkb}\|) = \mathcal{O}\left(h^{-|\beta|}\mathrm{e}^{-\frac{2S}{h}}\right).$$

Thus, Lemma 2.1.28 and (2.1.42) together with the Leibniz rule then imply

$$\left\|\partial^{\beta}_{z\bar{z}}(e_0(z) - e_{wkb}(z))\right\| = \mathcal{O}\left(h^{-|\beta|}\mathrm{e}^{-\frac{S}{h}}\right).$$

Part II - Now, let $z \in \Omega^{a}_{\eta}$ with $h^{\frac{2}{3}} \ll \eta <$ const. The statements of the proposition follow from a simple rescaling argument. For the rescaling we use the same notation as in the beginning of Section 2.1. Let $\widetilde{e}_0(\widetilde{z})$ be the $L^2(S^1/\sqrt{\eta}, d\widetilde{x})$-normalized eigenfunction of the operator $\widetilde{Q}(\widetilde{z}) = (\widetilde{P}_{\widetilde{h}} - \widetilde{z})^*(\widetilde{P}_{\widetilde{h}} - \widetilde{z})$ and note that $\eta^{\frac{1}{4}} e^{\eta}_{wkb}$ is $L^2(S^1/\sqrt{\eta}, d\widetilde{x})$-normalized. Thus,

$$\left\|\partial^{\beta}_{z\bar{z}}(\widetilde{e}_0(\widetilde{z}) - e^{\eta}_{wkb}(\cdot, \widetilde{z}, \widetilde{h})\right\|_{L^2(S^1/\sqrt{\eta}, d\widetilde{x})} = \mathcal{O}\left(\widetilde{h}^{-|\beta|}e^{-\frac{\widetilde{S}}{\widetilde{h}}}\right),$$

where $\widetilde{S}$ is as in (2.1.24). Since $e_0(z) = \eta^{-1/4}\widetilde{e}_0(\widetilde{z})$, it follows by rescaling that

$$\left\|\partial^{\beta}_{z\bar{z}}(e_0(z) - e^{\eta}_{wkb}(z))\right\|_{L^2(S^1, dx)} = \mathcal{O}\left(\eta^{\frac{|\beta|}{2}} h^{-|\beta|} e^{-\frac{S}{h}}\right).$$

The results on $\|\partial^{\beta}_{z\bar{z}} e^{\eta}_{wkb}\|$ and on $\|\partial^{\beta}_{z\bar{z}} e_0\|$ can be proven by the same rescaling argument. □

2.2 | Grushin problem for the unperturbed operator P_h

To start with we give a short refresher on Grushin problems since they have become an essential tool in microlocal analysis and it is a key method to the present work. As reviewed in [74], the central idea is to set up an auxiliary problem of the form

$$\begin{pmatrix} P(z) & R_- \\ R_+ & 0 \end{pmatrix} : \mathcal{H}_1 \oplus \mathcal{H}_- \longrightarrow \mathcal{H}_2 \oplus \mathcal{H}_+,$$

where $P(z)$ is the operator of interest and $R_\pm$ are suitably chosen. We say that the Grushin problem is well-posed if this matrix of operators is bijective. If $\dim \mathcal{H}_- = \dim \mathcal{H}_+ < \infty$, one usually writes

$$\begin{pmatrix} P(z) & R_- \\ R_+ & 0 \end{pmatrix}^{-1} = \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{pmatrix}.$$

The key observation, going back to the Shur complement formula or equivalently the Lyapunov-Schmidt bifurcation method, is that the operator $P(z) : \mathcal{H}_1 \to \mathcal{H}_2$ is invertible if and only if the finite dimensional matrix $E_{-+}(z)$ is invertible and when $E_{-+}(z)$ is invertible, we have

$$P^{-1}(z) = E(z) - E_+(z)E_{-+}^{-1}(z)E_-(z).$$

$E_{-+}(z)$ is sometimes called effective Hamiltonian.

The principal aim of this section is to introduce the three different Grushin Problems needed to study P_h^{δ}: one valid in all of Σ which is however less explicit (here we will follow the construction given in [67, Sec. 7.2 and 7.4]), and two very explicit Grushin Problems, one valid in the interior of Σ and one valid close to $\partial\Sigma$ (here we will recall the construction given by Hager in [32] respectively Bordeaux-Montrieux in [4]).

2.2.1 – Grushin problem valid in all of Σ

Following the ideas of [67], we will use the eigenfunctions e_0 and f_0 to set up the Grushin problem

Proposition 2.2.1. *Let $z \in \Omega \Subset \Sigma$ be open and relatively compact, and let α_0 be as in (1.2.8). Define*

$$\begin{aligned} R_+ &: H^1(S^1) \longrightarrow \mathbb{C} : \; u \longmapsto (u|e_0) \\ R_- &: \mathbb{C} \longrightarrow L^2(S^1) : \; u_- \longmapsto u_- f_0. \end{aligned} \tag{2.2.1}$$

Then

$$\mathscr{P}(z) := \begin{pmatrix} P_h - z & R_- \\ R_+ & 0 \end{pmatrix} : \; H^1(S^1) \times \mathbb{C} \longrightarrow L^2(S^1) \times \mathbb{C}$$

is bijective with the bounded inverse

$$\mathscr{E}(z) = \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{pmatrix}$$

where $E_-(z)v = (v|f_0)$, $E_+(z)v_+ = v_+ e_0$ and $E(z) = (P_h - z)^{-1}|_{(f_0)^\perp \to (e_0)^\perp}$ and $E_{-+}(z)v_+ = -\alpha_0 v_+$. Furthermore, we have the estimates

- *for $z \in \Omega$ with* $\mathrm{dist}(\Omega, \partial\Sigma) > 1/C$

$$\begin{aligned} &\|E_-(z)\|_{L^2 \to \mathbb{C}}, \|E_+(z)\|_{\mathbb{C} \to H^1} = \mathcal{O}(1), \\ &\|E(z)\|_{L^2 \to H^1} = \mathcal{O}(h^{-1/2}), \\ &|E_{-+}(z)| = \mathcal{O}\left(\sqrt{h}\mathrm{e}^{-\frac{S}{h}}\right) = \mathcal{O}\left(\mathrm{e}^{-\frac{1}{Ch}}\right); \end{aligned} \tag{2.2.2}$$

- *for $z \in \Omega \cap \Omega^a_\eta$ with $h^{\frac{2}{3}} \ll \eta <$* const.

$$\begin{aligned} &\|E_-(z)\|_{L^2 \to \mathbb{C}}, \|E_+(z)\|_{\mathbb{C} \to H^1} = \mathcal{O}(1), \\ &\|E(z)\|_{L^2 \to H^1} = \mathcal{O}((h\sqrt{\eta})^{-1/2}), \\ &|E_{-+}(z)| = \mathcal{O}\left(\sqrt{h}\eta^{\frac{1}{4}}\mathrm{e}^{-\frac{S}{h}}\right) = \mathcal{O}\left(\mathrm{e}^{-\frac{\asymp \eta^{3/2}}{h}}\right). \end{aligned} \tag{2.2.3}$$

Proof. For a proof of the existence of the bounded inverse as well as the estimate for $\|E(z)\|_{L^2 \to H^1}$ in the case of $\mathrm{dist}(\Omega, \partial\Sigma) > 1/C$ see [67, Section 7.2].

The other estimate for $\|E(z)\|_{L^2 \to H^1}$ can be proven by performing the same steps as in the case of $\mathrm{dist}(\Omega, \partial\Sigma) > 1/C$, mutatis mutandis, together with the estimate given by Bordeaux-Montrieux in [4, Proposition 4.3.5]. The estimates for $|E_{-+}(z)|$ follow from Proposition 2.1.7, whereas the estimates on $\|E_-(z)\|_{L^2 \to \mathbb{C}}$ and $\|E_+(z)\|_{\mathbb{C} \to H^1}$ come from the fact that e_0 and f_0 are normalized.

Alternatively, one can conclude the result in the case of $z \in \Omega \cap \Omega^a_\eta$ by a rescaling argument similar to the one in the proof of Proposition 2.1.11. □

2.2.2 – Tunneling

We prove now the following formula for a tunnel effect between e_0 which is microlocalized in $\rho_+(z)$ and f_0 which is microlocalized in $\rho_-(z)$ (cf. (1.1.14) and Proposition 2.2.7), from which we conclude Proposition 1.2.8. Recall in particular that S is the imaginary part of the action between $\rho_+(z)$ and $\rho_-(z)$ (cf. Definition 1.2.2).

Proposition 2.2.2. *Let $z \in \Omega \Subset \Sigma$ and let e_0 and f_0 be as in (1.2.6) and in (1.2.9). Furthermore, let $\Phi(z,h)$ be as in Proposition 1.2.5, let S be as in Definition 1.2.2 and let p be (1.1.7) as in and $\rho_\pm$ be as in (1.1.14). Let $h^{\frac{2}{3}} \ll \eta <$ const. Then, for all $z \in \Omega$ with $|\mathrm{Im}\, z - \langle \mathrm{Im}\, g \rangle| > 1/C$, $C \gg 1$,*

$$|(e_0|f_0)| = \frac{\left(\frac{i}{2}\{p,\overline{p}\}(\rho_+)\frac{i}{2}\{\overline{p},p\}(\rho_-)\right)^{\frac{1}{4}}}{\sqrt{\pi h}} |\partial_{\mathrm{Im}\, z} S(z)| \left(1 + \mathcal{O}\left(\eta^{-\frac{3}{4}} h^{\frac{1}{2}}\right)\right) \mathrm{e}^{-\frac{S}{h}}$$

where for all $\beta \in \mathbb{N}^2$

$$\partial^{\beta}_{z\overline{z}}\mathcal{O}\left(\eta^{-3/4}h^{\frac{1}{2}}\right) = \mathcal{O}\left(\eta^{\frac{|\beta|}{2}-\frac{3}{4}}h^{-|\beta|+\frac{1}{2}}\right).$$

This implies Proposition 1.2.8. Furthermore, Proposition 2.2.2 implies by direct calculation the following result:

Proposition 2.2.3. *Under the assumptions of Proposition 2.2.2 we have for* $h^{\frac{2}{3}} \ll \eta < const.$

$$\partial_{\operatorname{Im} z}|(e_0|f_0)|^2 = \frac{2\left(\frac{i}{2}\{p,\overline{p}\}(\rho_+)\frac{i}{2}\{\overline{p},p\}(\rho_-)\right)^{\frac{1}{2}}}{\pi h^2}|\partial_{\operatorname{Im} z}S(z)|^2(-\partial_{\operatorname{Im} z}S(z))\mathrm{e}^{-\frac{2S}{h}} + \mathcal{O}\left(\eta^{5/4}h^{-\frac{3}{2}}\mathrm{e}^{-\frac{2S}{h}}\right),$$

$$\partial_{\operatorname{Re} z}|(e_0|f_0)|^2, \partial_{\operatorname{Re} z}\partial_{\operatorname{Im} z}|(e_0|f_0)|^2 = \mathcal{O}\left(\mathrm{e}^{-\frac{1}{Ch}}\mathrm{e}^{-\frac{2S}{h}}\right).$$

Remark 2.2.4. Let us point out that we can find an even more detailed formula for $|(e_0|f_0)|$ (cf. (2.2.9)) valid even for $|\operatorname{Im} z - \langle \operatorname{Im} g\rangle| \geq 1/C$:

$$|(e_0|f_0)| = \frac{\left(\frac{i}{2}\{p,\overline{p}\}(\rho_+)\frac{i}{2}\{\overline{p},p\}(\rho_-)\right)^{\frac{1}{4}}}{\sqrt{\pi h}}\mathrm{e}^{-\frac{S}{h}}|\partial_{\operatorname{Im} z}S|\left(1+\frac{2\pi-|\partial_{\operatorname{Im} z}S|}{|\partial_{\operatorname{Im} z}S|}\mathrm{e}^{\operatorname{Re}\Phi}\right) + \mathcal{O}\left(\mathrm{e}^{-\frac{S}{h}}\right)+\mathcal{O}\left(\eta^{3/4}h^{-\frac{1}{2}}\mathrm{e}^{-\frac{S}{h}+\operatorname{Re}\Phi}\right)$$

Proof of Proposition 2.2.2. First, suppose that $z \in \Omega$ with $\operatorname{dist}(\Omega, \partial\Sigma) > 1/C$. Then, by Proposition 2.1.11

$$(e_0|f_0) = (e_0|f_{wkb}) + \mathcal{O}\left(\mathrm{e}^{-\frac{S}{h}}\right) = (e_{wkb}|f_{wkb}) + \mathcal{O}\left(\mathrm{e}^{-\frac{S}{h}}\right). \tag{2.2.4}$$

Recall the definition of the quasimodes e_{wkb} and f_{wkb} from Section 2.1. Moreover, recall from Section 1.1.1 that by the natural projection $\Pi : \mathbb{R} \to S^1$ we identify S^1 with the interval $[x_-(z) - 2\pi, x_-(z)[$. This choice leads to the fact that ϕ_+ is given by

$$\phi_+(x) = \int_{x_+(z)}^{x}(z-g(y))dy$$

on this interval, whereas ϕ_- is given by

$$\phi_-(x) = \begin{cases} \displaystyle\int_{x_-(z)}^{x}\overline{(z-g(y))}dy, & \text{for } x \in [x_+(z), x_-(z)[, \\ \displaystyle\int_{x_-(z)-2\pi}^{x}\overline{(z-g(y))}dy, & \text{for } x \in [x_-(z)-2\pi, x_+(z)[. \end{cases}$$

Define

$$R := \frac{a\overline{b}}{\sqrt{h}} = \frac{\left(\frac{i}{2}\{p,\overline{p}\}(\rho_+)\frac{i}{2}\{\overline{p},p\}(\rho_-)\right)^{\frac{1}{4}}}{\sqrt{\pi}} + \mathcal{O}(\sqrt{h}), \tag{2.2.5}$$

where we used Lemma 2.1.3, Proposition 2.1.4 and (2.2.22) to gain the equality. A straight forward computation yields that

$$(e_{wkb}|f_{wkb}) = R\mathrm{e}^{\frac{i}{h}\int_{x_+(z)}^{x_-(z)-2\pi}(z-g(y))dy}\int_{x_-(z)-2\pi}^{x_+(z)}\chi_e(x)\chi_f(x)dx + R\mathrm{e}^{\frac{i}{h}\int_{x_+(z)}^{x_-(z)}(z-g(y))dy}\int_{x_+(z)}^{x_-(z)}\chi_e(x)\chi_f(x)dx. \tag{2.2.6}$$

Using (2.1.6) and Definition 2.1.6, we have that

$$\begin{aligned}\int_{x_-(z)-2\pi}^{x_+(z)} \chi_e(x)\chi_f(x)dx &= x_+(z) - (x_-(z)-2\pi) \\ &\quad - \int_{x_- -2\pi}^{x_- -2\pi+\sqrt{h}} (1-\chi_e(x))dx - \int_{x_+ +2\pi-\sqrt{h}}^{x_+ +2\pi} (1-\chi_f(x))dx \\ &= x_+(z) - (x_-(z)-2\pi) + \mathcal{O}\left(\sqrt{h}\right), \end{aligned} \tag{2.2.7}$$

and similarly

$$\int_{x_+(z)}^{x_-(z)} \chi_e(x)\chi_f(x)dx = x_-(z) - x_+(z) + \mathcal{O}\left(\sqrt{h}\right). \tag{2.2.8}$$

Now let us assume that we are below the spectral line of P_h, i.e. $\operatorname{Im} z \le \langle \operatorname{Im} g\rangle$. There, we see that

$$\begin{aligned}|(e_{wkb}|f_{wkb})| = |R| \mathrm{e}^{-\frac{1}{h}\operatorname{Im}\int_{x_+(z)}^{x_-(z)}(z-g(y))dy} \Big| (x_-(z)-x_+(z)) + \mathcal{O}\left(\sqrt{h}\right) \\ + \left(x_+(z) - (x_-(z)-2\pi) + \mathcal{O}\left(\sqrt{h}\right)\right) \mathrm{e}^{-\frac{2\pi i}{h}(z-\langle g\rangle)}\Big|.\end{aligned}$$

Analogously, if we are above the spectral line, i.e. $\operatorname{Im} z \ge \langle \operatorname{Im} g\rangle$,

$$\begin{aligned}|(e_{wkb}|f_{wkb})| = |R| \mathrm{e}^{-\frac{1}{h}\operatorname{Im}\int_{x_+(z)}^{x_-(z)-2\pi}(z-g(y))dy} \Big| (x_+(z) - (x_+(z)-2\pi)) \\ + \mathcal{O}\left(\sqrt{h}\right) + \left(x_-(z) - x_+(z) + \mathcal{O}\left(\sqrt{h}\right)\right) \mathrm{e}^{\frac{2\pi i}{h}(z-\langle g\rangle)}\Big|.\end{aligned}$$

Together with (2.2.4), we conclude that

$$\begin{aligned}|(e_0|f_0)| = \frac{\left(\frac{i}{2}\{p,\overline{p}\}(\rho_+)\frac{i}{2}\{\overline{p},p\}(\rho_-)\right)^{\frac{1}{4}}}{\sqrt{\pi h}} \mathrm{e}^{-\frac{S}{h}} |\partial_{\operatorname{Im} z} S| \left(1 + \frac{2\pi - |\partial_{\operatorname{Im} z} S|}{|\partial_{\operatorname{Im} z} S|}\mathrm{e}^{\operatorname{Re}\Phi}\right) \\ + \mathcal{O}\left(\mathrm{e}^{-\frac{S}{h}}\right) + \mathcal{O}\left(\eta^{3/4} h^{-\frac{1}{2}} \mathrm{e}^{-\frac{S}{h}+\operatorname{Re}\Phi}\right)\end{aligned} \tag{2.2.9}$$

where $\Phi = \Phi(z,h)$ is as in Proposition 1.2.5. Note that $\exp\{\Phi(z,h)\}$ is exponentially small for $|\operatorname{Im} z - \langle \operatorname{Im} g\rangle| > 1/C$. Thus,

$$|(e_0|f_0)| = \frac{\left(\frac{i}{2}\{p,\overline{p}\}(\rho_+)\frac{i}{2}\{\overline{p},p\}(\rho_-)\right)^{\frac{1}{4}}}{\sqrt{\pi h}} \mathrm{e}^{-\frac{S}{h}} |\partial_{\operatorname{Im} z} S(z)| \left(1 + \mathcal{O}\left(\eta^{-3/4} h^{\frac{1}{2}}\right)\right). \tag{2.2.10}$$

Now let us discuss the $\partial^{\beta}_{z\overline{z}}$-derivatives of the errors. First let us treat the error term $\mathcal{O}\left(\sqrt{h}\right)$ from the definition of R which is given as a product of the normalization coefficients of the quasimodes e_{wkb} and f_{wkb}. Thus, it is easy to see that

$$\partial^{\beta}_{z\overline{z}} \mathcal{O}\left(\sqrt{h}\right) = \mathcal{O}\left(h^{-(|\beta|-1/2)}\right). \tag{2.2.11}$$

The $\partial^{\beta}_{z\overline{z}}$-derivatives of the error term in (2.2.7), (2.2.8) can be treated as follows: note that

$$\begin{aligned}&\partial_z \int_{x_- -2\pi}^{x_- -2\pi+\sqrt{h}} (1-\chi_e(x,z))dx = \\ &\left(\chi_e(x_- -2\pi, z) - \chi_e(x_- -2\pi+\sqrt{h}, z)\right)\partial_z x_- - \int_{x_- -2\pi}^{x_- -2\pi+\sqrt{h}} \partial_z \chi_e(x,z)dx.\end{aligned}$$

By (2.1.16)

$$\begin{aligned}\int_{x_- -2\pi}^{x_- -2\pi+\sqrt{h}} \partial_z \chi_e(x,z)dx &= -\int_{x_- -2\pi}^{x_- -2\pi+\sqrt{h}} \psi\left(\frac{x - x_- + 2\pi}{\sqrt{h}}\right) \partial_z x_-(z) dx \\ &= -\partial_z x_-(z).\end{aligned}$$

Since $\chi_e(x_- - 2\pi, z) = 0$ and $\chi_e(x_- - 2\pi + \sqrt{h}, z) = 1$,

$$\partial_z \int_{x_- - 2\pi}^{x_- - 2\pi + \sqrt{h}} (1 - \chi_e(x,z))dx = 0.$$

(2.2.8) as well as the respective $\overline{z}$-derivatives can be treated analogously, and we conclude that $\partial^\beta_{z\overline{z}}\mathcal{O}(\sqrt{h}) = 0$ for all $\beta \in \mathbb{N}^2 \backslash \{0\}$. Hence, we have

$$\partial_z^n \partial_{\overline{z}}^m \mathcal{O}\left(\eta^{-3/4} h^{\frac{1}{2}}\right) = \mathcal{O}\left(\eta^{\frac{|\beta|}{2} - \frac{3}{4}} h^{-|\beta| + \frac{1}{2}}\right).$$

Finally, in the case where $z \in \Omega \cap \Omega^a_\eta$ we can conclude the statement by a rescaling argument similar as in the proof of Proposition 2.1.11. □

Remark 2.2.5. It is a direct consequence of (2.2.6), (2.2.4) and Proposition 2.1.11 that

$$\partial^\beta_{z\overline{z}}(e_0|f_0) = \mathcal{O}\left(\eta^{\frac{|\beta|+3/2}{2}} h^{-(|\beta|+1/2)} \mathrm{e}^{-\frac{S}{h}}\right),$$

where we conclude the case where $z \in \Omega \cap \Omega^a_\eta$ by a rescaling argument similar as in the proof of Proposition 2.1.11.

Proof of Proposition 2.2.3. The first statement follows directly from Proposition 2.2.2. The statements regarding the derivatives can be derived by a direct calculation from Proposition 2.2.2 together with the fact that the z- respectively the $\overline{z}$-derivative of the error term increases its growth at most by a term of order $\eta^{1/2}h^{-1}$. Moreover, we use that e^{Φ} is exponentially small in h due to $|\mathrm{Im}\, z - \langle \mathrm{Im}\, g\rangle| > 1/C$. Furthermore, we use that the prefactor $\left(\frac{i}{2}\{p,\overline{p}\}(\rho_+)\frac{i}{2}\{\overline{p},p\}(\rho_-)\right)^{\frac{1}{4}}$ is the first order term of R (cf. (2.2.5)). Recall that R is defined via the normalization coefficients of the quasi-modes e_{wkb} and f_{wkb}. It is thus independent of $\mathrm{Re}\, z$ and its $\partial_{\mathrm{Im}\, z}$ derivative is of order $\mathcal{O}(\eta^{-1/4})$ which can be seen by the stationary phase method and a rescaling argument similar to the one in the proof of Proposition 2.1.11. □

Now let us give estimates on the derivatives of the effective Hamiltonian $E_{-+}(z)$.

Proposition 2.2.6. *Let $z \in \Omega \Subset \Sigma$ and let $E_{-+}(z)$ be as in Proposition 2.2.1. Then there exists a $C > 0$ such that for $h > 0$ small enough and all $\beta \in \mathbb{N}^2$*

$$|\partial^\beta_{z\overline{z}} E_{-+}(z)| = \mathcal{O}\left(\eta^{\frac{|\beta|+1/2}{2}} h^{-|\beta|+1/2} \mathrm{e}^{-\frac{S}{h}}\right).$$

Proof. Take the $\partial_{\overline{z}}$ derivative and the ∂_z derivative of the first equation in (1.2.8) to gain

$$(P_h - z)\partial_{\overline{z}} e_0 = (\partial_{\overline{z}}\alpha_0) f_0 + \alpha_0 \partial_{\overline{z}} f_0, \ (P_h - z)\partial_z e_0 - e_0 = (\partial_z \alpha_0) f_0 + \alpha_0 \partial_z f_0.$$

Now consider the scalar product of these equations with f_0 and recall from Proposition 2.2.1 that $E_{-+}(z) = -\alpha_0(z)$ to conclude

$$\begin{aligned} &\partial_{\overline{z}} E_{-+}(z) = E_{-+}(z)\left\{(\partial_{\overline{z}} e_0|e_0) - (\partial_{\overline{z}} f_0|f_0)\right\} \text{ and} \\ &\partial_z E_{-+}(z) = E_{-+}(z)\left\{(\partial_z e_0|e_0) - (\partial_z f_0|f_0)\right\} + (e_0|f_0). \end{aligned} \tag{2.2.12}$$

The statement of the Proposition then follows by repeated differentiation of (2.2.12) and induction using Remark 2.2.5, the estimate $|E_{-+}(z)| = \mathcal{O}(\eta^{\frac{1}{4}} h^{\frac{1}{2}} \mathrm{e}^{-\frac{S}{h}})$ given in (2.2.2) and (2.2.3) and the estimates given in Proposition 2.1.11. □

Finally, Proposition 2.2.2 permits us to prove the following extension of Proposition 2.1.11:

Proposition 2.2.7. *Let $z \in \Omega \Subset \Sigma$ and let e_0 and f_0 be the eigenfunctions of the operators Q and $\tilde{Q}$ with respect to their smallest eigenvalue (as in Section 2.2.1). Let $S = S(z)$ be defined as in Definition 1.2.2. Then*

- *for $z \in \Omega$ with* $\operatorname{dist}(\Omega, \partial\Sigma) > 1/C$ *and for all $\alpha \in \mathbb{N}^3$*

$$\|\partial^\alpha_{z\bar{z}x}(e_0 - e_{wkb})\|, \ \|\partial^\alpha_{z\bar{z}x}(f_0 - f_{wkb})\| = \mathcal{O}\left(h^{-|\alpha|} e^{-\frac{S}{h}}\right).$$

Here, we set $\partial^\alpha_{z\bar{z}x} = \partial_z^{\alpha_1}\partial_{\bar{z}}^{\alpha_2}\partial_x^{\alpha_3}$. Furthermore, the various z-, $\bar{z}$- and x-derivatives of e_0, f_0, e_{wkb} and f_{wkb} have at most temperate growth in $1/h$, more precisely

$$\|\partial^\alpha_{z\bar{z}x} e_{wkb}\|, \ \|\partial^\alpha_{z\bar{z}x} f_{wkb}\|, \ \|\partial^\alpha_{z\bar{z}x} e_0\|, \ \|\partial^\alpha_{z\bar{z}x} f_0\| = \mathcal{O}\left(h^{-|\alpha|}\right)$$

for all $\alpha \in \mathbb{N}^3$;

- *for $h^{2/3} \ll \eta < const.$, $z \in \Omega \cap \Omega^a_\eta$ and for all $\alpha \in \mathbb{N}^3$*

$$\|\partial^\alpha_{z\bar{z}x}(e_0 - e^\eta_{wkb})\|, \ \|\partial^\alpha_{z\bar{z}x}(f_0 - f^\eta_{wkb})\| = \mathcal{O}\left(\eta^{\frac{\alpha_1+\alpha_2}{2}+\alpha_3} h^{-|\alpha|} e^{-\frac{S}{h}}\right).$$

Furthermore, the various z-, $\bar{z}$- and x-derivatives of e_0, f_0, e^η_{wkb} and f^η_{wkb} have at most temperate growth in $\sqrt{\eta}/h$, more precisely

$$\|\partial^\alpha_{z\bar{z}x} e^\eta_{wkb}\|, \ \|\partial^\alpha_{z\bar{z}x} f^\eta_{wkb}\|, \ \|\partial^\alpha_{z\bar{z}x} e_0\|, \ \|\partial^\alpha_{z\bar{z}x} f_0\| = \mathcal{O}\left(\eta^{\frac{\alpha_1+\alpha_2}{2}+\alpha_3} h^{-|\alpha|}\right)$$

for all $\alpha \in \mathbb{N}^3$.

Proof. Will show the proof in the case of $e_0(z)$ since the case of $f_0(z)$ is similar. Suppose first that $z \in \Omega$ with $\operatorname{dist}(\Omega, \partial\Sigma) > 1/C$. Recall from (1.2.8) that

$$(P_h - z)e_0 = \alpha_0 f_0 \quad \text{and} \quad (P_h - z)^* f_0 = \bar{\alpha}_0 e_0 \tag{2.2.13}$$

First consider the $\partial_z^n \partial_{\bar{z}}^m$ derivatives of (2.2.13):

$$(P_h - z)\partial_z^n\partial_{\bar{z}}^m e_0(z) = n\partial_z^{n-1}\partial_{\bar{z}}^m e_0(z) + \sum_{\substack{|\alpha_1+\beta_1|=n \\ |\alpha_2+\beta_2|=m}} \binom{\eta+\beta}{\beta} (\partial^\eta \alpha_0(z))(\partial^\beta f_0(z)) \tag{2.2.14}$$

and

$$(P_h - z)^*\partial_z^n\partial_{\bar{z}}^m f_0(z) = m\partial_z^n\partial_{\bar{z}}^{m-1} f_0(z) + \sum_{\substack{|\alpha_1+\beta_1|=n \\ |\alpha_2+\beta_2|=m}} \binom{\eta+\beta}{\beta} (\partial^\eta \bar{\alpha}_0(z))\partial^\beta e_0(z)$$

and thus

$$\begin{aligned} h\|D_x\partial_z^n\partial_{\bar{z}}^m e_0(z)\| \leq & n\|\partial_z^{n-1}\partial_{\bar{z}}^m e_0(z)\| + \sum_{\substack{|\alpha_1+\beta_1|=n \\ |\alpha_2+\beta_2|=m}} \binom{\eta+\beta}{\beta} \|\partial^\eta \alpha_0(z)\|\|\partial^\beta f_0(z)\| \\ & + \|g - z\|_{L^\infty(S^1)} \cdot \|\partial_z^n\partial_{\bar{z}}^m e_0(z)\| \end{aligned}$$

and

$$\begin{aligned} h\|D_x\partial_z^n\partial_{\bar{z}}^m f_0(z)\| \leq & m\|\partial_z^n\partial_{\bar{z}}^{m-1} f_0(z)\| + \sum_{\substack{|\alpha_1+\beta_1|=n \\ |\alpha_2+\beta_2|=m}} \binom{\eta+\beta}{\beta} \|\partial^\eta \bar{\alpha}_0(z)\|\|\partial^\beta e_0(z)\| \\ & + \|g - z\|_{L^\infty(S^1)} \cdot \|\partial_z^n\partial_{\bar{z}}^m f_0(z)\|. \end{aligned}$$

By Proposition 2.2.6, there exists a constant $C > 0$ such that

$$|\partial_z^k\partial_{\bar{z}}^j \alpha_0(z)| = |\partial_z^k\partial_{\bar{z}}^j E_{-+}(z)| = \mathcal{O}\left(h^{-(k+j)} \mathrm{e}^{-\frac{S}{h}}\right). \tag{2.2.15}$$

By (2.1.28) we conclude

$$\|D_x\partial_z^n\partial_{\bar z}^m e_0(z)\|,\ \|D_x\partial_z^n\partial_{\bar z}^m f_0(z)\| = \mathcal{O}\left(h^{-(n+m+1)}\right).$$

Repeated differentiation of (2.2.14) and induction then yield that for all $l \in \mathbb{N}$

$$\|D_x^l\partial_z^n\partial_{\bar z}^m e_0(z)\|, \|D_x^l\partial_z^n\partial_{\bar z}^m f_0(z)\| = \mathcal{O}\left(h^{-(l+n+m)}\right).$$

The estimate

$$\|D_x^l\partial_z^n\partial_{\bar z}^m e_{wkb}\|, \|D_x^l\partial_z^n\partial_{\bar z}^m f_{wkb}\| = \mathcal{O}\left(h^{-(l+n+m)}\right)$$

follows directly by the stationary phase method together with (2.1.9), (2.1.16). Finally, using (1.2.8), (2.1.7), consider

$$(P_h - z)(e_0 - e_{wkb}) = \alpha_0 f_0 - h^{-\frac{1}{4}} a(z)\frac{h}{i}\partial_x\chi_e \mathrm{e}^{\frac{i}{h}\phi_+(x)}$$

which implies for $k \geq 1$ that $(hD_x)^k\partial_z^n\partial_{\bar z}^m(e_0 - e_{wkb})$ is equal to

$$(hD_x)^{(k-1)}\partial_z^n\partial_{\bar z}^m(\alpha_0 f_0) - (hD_x)^{(k-1)}\partial_z^n\partial_{\bar z}^m\left(h^{-\frac{1}{4}} a(z)\frac{h}{i}\partial_x\chi_e\mathrm{e}^{\frac{i}{h}\phi_+(x)}\right) + (hD_x)^{(k-1)}\partial_z^n\partial_{\bar z}^m(g(x) - z)(e_0 - e_{wkb}).$$

By induction over k together with Proposition 2.1.11 and (2.2.15), (2.1.16), we conclude the first point of the Proposition. The results in the case where $z \in \Omega \cap \Omega_\eta^a$ follow by a rescaling argument similar as in the proof of Proposition 2.1.11. □

2.2.3 – Alternative Grushin problems for the unperturbed operator P_h

In [32] Hager set up a different Grushin problem for P_h and $z \in \Omega_i$ which results in a more explicit effective Hamiltonian $E_{-+}^H(z)$. To avert confusion, we will mark the elements of Hager's Grushin problem with an additional "H".

Bordeaux-Montrieux in [4] then extended Hager's Grushin problem to $z \in \Omega \cap \Omega_\eta^a$. It is very useful for the further discussion to have an explicit effective Hamiltonian. Thus we will briefly introduce Hager's Grushin problem $\mathscr{P}^H$ and show that $E_{-+}(z)$ and $E_{-+}^H(z)$ differ only by an exponentially small error.

Proposition 2.2.8 ([32, 4])**.** *For $z \in \Omega \Subset \Sigma$, let $x_\pm(z) \in \mathbb{R}$ be as in (1.1.14).*

- *for $z \in \Omega$ with $\operatorname{dist}(\Omega, \partial\Sigma) > 1/C$: let $I_\pm$ be open intervals, independent of z such that*

$$x_\pm(z) \in I_\pm,\ x_\mp(z) \notin \overline{I_\pm} \quad \textit{for all } z \in \overline{\Omega}.$$

Let $\phi_\pm(x,z)$ be as in Definition 2.1.2. Then, there exist smooth functions $c_\pm(z;h) > 0$ such that

$$c_\pm(z;h) \sim h^{-\frac{1}{4}}\left(c_\pm^0(z) + hc_\pm^1(z) + \dots\right)$$

and, for $e_+(z;h) := c_+(z;h)\exp(\frac{i\phi_+(x,z)}{h}) \in H^1(I_+)$ and $e_-(z;h) := c_-(z;h)\exp(\frac{\overline{i\phi_-(x,z)}}{h}) \in H^1(I_+)$,

$$\|e_+\|_{L^2(I_+)} = 1 = \|e_-\|_{L^2(I_-)}.$$

Furthermore, we have

$$c_+^0(z) = \left(\frac{-\operatorname{Im} g'(x_+(z))}{\pi}\right)^{\frac{1}{4}},\ \textit{and } c_-^0(z) = \left(\frac{\operatorname{Im} g'(x_-(z))}{\pi}\right)^{\frac{1}{4}}.$$

- *for $z \in \Omega \cap \Omega_\eta^a$ with $h^{2/3} \ll \eta < const.$: let $J_\pm$ be open intervals, such that*

$$x_\pm(\Omega_\eta^a) \in J_\pm, \quad \operatorname{dist}(J_+, J_-) > \frac{1}{C}\eta^{1/2}.$$

Define $\tilde{I}_\pm := S^1 \backslash \overline{J_\mp}$. Let $\phi_\pm(x,z)$ be as in Definition 2.1.2 and set $\tilde{h} := h/\eta^{3/2}$. Then, there exist smooth functions $c_\pm(z;\tilde{h}) > 0$ such that

$$c_\pm^\eta(z;\tilde{h}) \sim \tilde{h}^{-\frac{1}{4}}\eta^{-1/4}\left(c_\pm^{0,\eta}(z) + \tilde{h}c_\pm^{1,\eta}(z) + \dots\right)$$

and, for $e_+^\eta(z;h) := c_+^\eta(z;\tilde{h})\exp(\frac{i\phi_+(x,z)}{h}) \in H^1(\tilde{I}_+)$ and $e_-^\eta(z;h) := c_-^\eta(z;\tilde{h})\exp(\frac{\overline{i\phi_-(x,z)}}{h}) \in H^1(\tilde{I}_+)$,

$$\|e_+^\eta\|_{L^2(\tilde{I}_+)} = 1 = \|e_-^\eta\|_{L^2(\tilde{I}_-)}.$$

Furthermore, we have

$$c_+^{0,\eta}(z) = \left(\frac{|\operatorname{Im} g''(a)(\tilde{x}_+(\tilde{z}) - a/\sqrt{\eta})(1+o(1))|}{\pi}\right)^{\frac{1}{4}}, \quad z \in \Omega_\eta^a,$$

$$c_-^{0,\eta}(z) = \left(\frac{|\operatorname{Im} g''(a)(\tilde{x}_-(\tilde{z}) - a/\sqrt{\eta})(1+o(1))|}{\pi}\right)^{\frac{1}{4}}, \quad z \in \Omega_\eta^a.$$

Proof. For a proof of the first statement see [32]. The second statement has been proven in [4] with the exception of the representation of $c_\pm^{0,\eta}(z)$ which can be achieved by an analogous argument to the one used in the proof of Proposition 2.1.4. □

Note that $(P_h - z)e_+^\bullet(x,z) = 0$ on I_+ and that $(P_h - z)^* e_-^\bullet(x,z) = 0$ on I_-. With these quasimodes Hager and then Bordeaux-Montrieux set up a Grushin problem $\mathscr{P}^H$ and proved the existence of an inverse $\mathscr{E}^H$.

Proposition 2.2.9 ([32])**.** *For $z \in \Omega_i \Subset \mathring{\Sigma}$ and $x_\pm(z)$ as in (1.1.14). Let $g \in \mathscr{C}^\infty(S^1;\mathbb{C})$ be as in (1.1.6) and let $a < b < a+2\pi$ where a denotes the minimum and b the maximum of* $\operatorname{Im} g$. *Let $J_+ \subset (b, a+2\pi)$ and $J_- \subset (a,b)$ such that $\overline{\{x_\pm(z) : z \in \Omega\}} \subset J_\pm$. Let $\chi_\pm \in \mathscr{C}_c^\infty(I_\pm)$ be such that $\chi_\pm \equiv 1$ on $\overline{J_\pm}$ and* $\operatorname{supp}(\chi_+) \cap \operatorname{supp}(\chi_-) = \emptyset$. *Define*

$$R_+^H : H^1(S^1) \longrightarrow \mathbb{C} : \; u \longmapsto (u|\chi_+ e_+)$$

$$R_-^H : \mathbb{C} \longrightarrow L^2(S^1) : \; u_- \longmapsto u_- \chi_- e_-.$$

Then

$$\mathscr{P}^H(z) := \begin{pmatrix} P_h - z & R_-^H \\ R_+^H & 0 \end{pmatrix} : \; H^1(S^1) \times \mathbb{C} \longrightarrow L^2(S^1) \times \mathbb{C}$$

is bijective with the bounded inverse

$$\mathscr{E}^H(z) = \begin{pmatrix} E^H(z) & E_+^H(z) \\ E_-^H(z) & E_{-+}^H(z) \end{pmatrix}$$

where

$$\|E^H(z)\|_{L^2 \to H^1} = \mathscr{O}(h^{-1/2}), \qquad \|E_-^H(z)\|_{L^2 \to \mathbb{C}} = \mathscr{O}(1),$$

$$\|E_+^H(z)\|_{\mathbb{C} \to H^1} = \mathscr{O}(1), \qquad |E_{-+}^H(z)| = \mathscr{O}\left(\mathrm{e}^{-\frac{1}{Ch}}\right). \tag{2.2.16}$$

Furthermore,

$$E_{-+}^H(z) = \left(\left(\frac{i}{2}\{p,\overline{p}\}(\rho_+)\frac{i}{2}\{\overline{p},p\}(\rho_-)\right)^{\frac{1}{4}}\left(\frac{h}{\pi}\right)^{\frac{1}{2}} + \mathscr{O}\left(h^{\frac{3}{2}}\right)\right) \cdot \left(\mathrm{e}^{\frac{i}{h}\int_{x_+}^{x_-}(z-g(y))dy} - \mathrm{e}^{\frac{i}{h}\int_{x_+}^{x_-+2\pi}(z-g(y))dy}\right), \tag{2.2.17}$$

where the prefactor of the exponentials depends only on $\operatorname{Im} z$ *and has bounded derivatives of order* $\mathscr{O}(\sqrt{h})$.

Proof. See [32]. □

Proposition 2.2.10 ([4]). *Let $\Omega \Subset \Sigma$. For $z \in \Omega\cap\Omega_\eta^{a,b}$ and $x_\pm(z)$ as in (1.1.14). Let $g \in \mathscr{C}^\infty(S^1)$ be as in (1.1.6). Let $J_\pm$ and $I_\pm$ be as in the second point of Proposition 2.2.8. Let $\chi_\pm^\eta \in \mathscr{C}_c^\infty(I_\pm)$ such that $\chi_\pm^\eta \equiv 1$ on $\overline{J_\pm}$ and $\operatorname{supp}(\chi_+^\eta)\cap\operatorname{supp}(\chi_-^\eta) = \emptyset$. Define*

$$R_+^\eta : H^1(S^1) \longrightarrow \mathbb{C} : \ u \longmapsto (u|\chi_+ e_+^\eta)$$
$$R_-^\eta : \mathbb{C} \longrightarrow L^2(S^1) : \ u_- \longmapsto u_-\chi_- e_-^\eta.$$

Then

$$\mathscr{P}^\eta(z) := \begin{pmatrix} P_h - z & R_-^\eta \\ R_+^\eta & 0 \end{pmatrix} : \ H^1(S^1)\times\mathbb{C} \longrightarrow L^2(S^1)\times\mathbb{C}$$

is bijective with the bounded inverse

$$\mathscr{E}^\eta(z) = \begin{pmatrix} E^\eta(z) & E_+^\eta(z) \\ E_-^\eta(z) & E_{-+}^\eta(z) \end{pmatrix}$$

where

$$\|E^\eta(z)\|_{L^2\to H^1} = \mathscr{O}((\sqrt{\eta}h)^{-1/2}), \qquad \|E_-^\eta(z)\|_{L^2\to\mathbb{C}} = \mathscr{O}(1),$$
$$\|E_+^\eta(z)\|_{\mathbb{C}\to H^1} = \mathscr{O}(1), \qquad |E_{-+}^\eta(z)| = \mathscr{O}\left(\eta^{1/4}h^{1/2}\mathrm{e}^{-\frac{\asymp\eta^{3/2}}{h}}\right). \tag{2.2.18}$$

Furthermore,

$$\begin{aligned} E_{-+}^\eta(z) = &\left(c_+^{0,\eta}(z)c_-^{0,\eta}(z)\left(h\sqrt{\eta}\right)^{\frac{1}{2}} + \mathscr{O}\left(h^{\frac{3}{2}}\eta^{-5/4}\right)\right) \\ &\cdot\left(\mathrm{e}^{\frac{i}{h}\int_{x_+}^{x_-}(z-g(y))dy} - \mathrm{e}^{\frac{i}{h}\int_{x_+}^{x_-+2\pi}(z-g(y))dy}\right), \end{aligned} \tag{2.2.19}$$

where the prefactor of the exponentials depends only on $\operatorname{Im} z$ *and has bounded derivatives of order* $\mathscr{O}(\sqrt{h\sqrt{\eta}})$.

Proof. See [4]. (2.2.19) has not been stated in this form on [4]. However, it can easily be deduce from the results in [4] together with Proposition 2.2.8. □

Remark 2.2.11. The cut-off function $\chi_\pm^\eta$ in the above proposition can be chosen similarly to $\chi_{e,f}^\eta$ in Definition 2.1.6 (compare also with Definition 2.1.2).

2.2.4 – Estimates on the effective Hamiltonians

In [32] Hager chose to represent S^1 as an interval between two of the periodically appearing minima of $\operatorname{Im} g$ and thus chose the notation for $x_\pm$ accordingly (this notation was used in (2.2.17)). In our case however, we chose to represent S^1 as an interval between two of the periodically appearing maxima of $\operatorname{Im} g$. This results in the following difference between notations:

$$x_+(z) = x_+^H(z) - 2\pi \quad \text{and} \quad x_-(z) = x_-^H(z).$$

Thus, in our notation, we have for $\bullet = H, \eta$

$$E_{-+}^\bullet(z) = V^\bullet(z,h)\left(\mathrm{e}^{\frac{i}{h}\int_{x_+}^{x_--2\pi}(z-g(y))dy} - \mathrm{e}^{\frac{i}{h}\int_{x_+}^{x_-}(z-g(y))dy}\right), \tag{2.2.20}$$

where $V^\bullet = V^\bullet(z,h)$ satisfies

$$V^\bullet = \begin{cases} \left(\frac{i}{2}\{p,\overline{p}\}(\rho_+)\frac{i}{2}\{\overline{p},p\}(\rho_-)\right)^{\frac{1}{4}}\left(\frac{h}{\pi}\right)^{\frac{1}{2}}(1+\mathscr{O}(h)), \text{if } \bullet = H, \ z\in\Omega_i \\ c_+^{0,\eta}(z)c_-^{0,\eta}(z)\left(h\sqrt{\eta}\right)^{\frac{1}{2}}\left(1+\mathscr{O}\left(\eta^{-\frac{3}{2}}h\right)\right), \text{if } \bullet = \eta, \ z\in\Omega_\eta^a. \end{cases} \tag{2.2.21}$$

Note that Taylor expansion around the point a yields

$$\begin{aligned}\{p,\overline{p}\}(\rho_\pm) &= -2i\operatorname{Im} g'(x_\pm) \\ &= 2i\sqrt{\eta}\left|\operatorname{Im} g''(a)(\tilde{x}_\pm(\tilde{z}) - a/\sqrt{\eta})(1+o_{\sqrt{\eta}}(1))\right|, \text{ for } z\in\Omega^a_\eta.\end{aligned} \tag{2.2.22}$$

Therefore, we may write for all $z\in\Omega\Subset\Sigma$

$$V(z,h) := V^\bullet(z,h) = \left(\frac{i}{2}\{p,\overline{p}\}(\rho_+)\frac{i}{2}\{\overline{p},p\}(\rho_-)\right)^{\frac{1}{4}}\left(\frac{h}{\pi}\right)^{\frac{1}{2}}\left(1+\mathcal{O}\left(\eta^{-\frac{3}{2}}h\right)\right) \tag{2.2.23}$$

where the first order term is $\eta^{1/4}$ for $z\in\Omega\cap\Omega^a_\eta$. Note that

$$\left|\mathrm{e}^{\frac{i}{h}\int_{x_+}^{x_--2\pi}(z-g(y))dy} - \mathrm{e}^{\frac{i}{h}\int_{x_+}^{x_-}(z-g(y))dy}\right| = \mathrm{e}^{-\frac{S}{h}}\left|1-\mathrm{e}^{\Phi(z,h)}\right|, \tag{2.2.24}$$

where $\Phi(z,h)$ is defined already in Proposition 1.2.5. For the readers convenience:

$$\Phi(z,h) = \begin{cases} -\frac{2\pi i}{h}(z-\langle g\rangle), \text{ if } \operatorname{Im} z < \langle \operatorname{Im} g\rangle, \\ \frac{2\pi i}{h}(z-\langle g\rangle), \text{ if } \operatorname{Im} z > \langle \operatorname{Im} g\rangle, \end{cases}$$

Hence

$$|E^\bullet_{-+}(z)| = V(z,h)\mathrm{e}^{-\frac{S}{h}}\left|1-\mathrm{e}^{\Phi(z,h)}\right|. \tag{2.2.25}$$

The aim of this section is to prove the following proposition.

Proposition 2.2.12. *Let $z\in\Omega\Subset\Sigma$, let $\Phi(z,h)$ be as in Proposition 1.2.5 and let $E_{-+}(z)$ the effective Hamiltonian given in Proposition 2.2.1. Then, for $h>0$ small enough, there exists a constant $C>0$ such that for $h^{\frac{2}{3}} \ll \eta \le$ const.*

$$|E_{-+}(z)| = V(z,h)\mathrm{e}^{-\frac{S(z)}{h}}\left|1-\mathrm{e}^{\Phi(z,h)}\right|\left(1+\mathcal{O}\left(\mathrm{e}^{-\frac{\asymp\eta^{3/2}}{h}}\right)\right).$$

Furthermore, for all $\beta\in\mathbb{N}^2$ the $\partial^\beta_{z\overline{z}}$ derivatives of the error terms are bounded and of order

$$\mathcal{O}\left(\eta^{\frac{|\beta|}{2}}h^{-|\beta|}\mathrm{e}^{-\frac{\asymp\eta^{\frac{3}{2}}}{h}}\right).$$

Proof of Proposition 1.2.9. Recall that $(P_h-z)e_0 = \alpha_0 f_0$ (cf. (1.2.8)). Suppose first that $z\in\Omega$ with $\operatorname{dist}(\Omega,\partial\Sigma) > 1/C$. By Proposition 2.1.11 we find

$$\begin{aligned}\left((1-\chi)(P_h-z)e_0|f_0\right) &= \alpha_0(f_0|(1-\chi)f_0) \\ &= \alpha_0\left((f_{wkb}|(1-\chi)f_{wkb}) + \mathcal{O}\left(\mathrm{e}^{-\frac{S}{h}}\right)\right).\end{aligned}$$

Since the phase of f_{wkb} has no critical point on the support of χ, it follows that there exists a constant $C>0$, depending on χ but uniform in $z\in\Omega$, such that

$$\left((1-\chi)(P_h-z)e_0|f_0\right) = \mathcal{O}\left(\alpha_0\mathrm{e}^{-\frac{1}{Ch}}\right).$$

By a similar argument we find that

$$\left((P_h-z)\chi e_0|f_0\right) = \alpha_0\left(\chi e_0|e_0\right) = \mathcal{O}\left(\alpha_0\mathrm{e}^{-\frac{1}{Ch}}\right).$$

In the case where $z\in\Omega\cap\Omega^a_\eta$, we perform a rescaling argument similar to the one in the proof of Proposition 2.1.11. Thus,

$$\left((1-\chi)(P_h-z)e_0|f_0\right), \left((P_h-z)\chi e_0|f_0\right) = \mathcal{O}\left(\alpha_0\exp\left\{-\frac{\eta^{\frac{3}{2}}}{Ch}\right\}\right).$$

Note that Proposition 2.1.11 implies that each z- and $\overline{z}$- derivative of the exponentially small error term increases its order of growth at most by factor of order $\mathcal{O}(\eta^{1/2}h^{-1})$. Thus, using (1.2.8) yields

$$\alpha_0 = \big((1-\chi+\chi)(P_h - z)e_0|f_0\big) = \big([\chi, P_h]e_0|f_0\big) + \mathcal{O}\left(\alpha_0 \exp\left\{-\frac{\eta^{\frac{3}{2}}}{Ch}\right\}\right) \tag{2.2.26}$$

The statement of the Proposition then follows by the fact that $|\alpha_0| = |E_{-+}(z)|$ (cf. Proposition 2.2.1) together with Proposition 2.2.12. □

We give some estimates on the elements of the Grushin problems introduced in Section 2.2.

Proposition 2.2.13. *Let $\Omega \Subset \Sigma$, let $E^\bullet_{-+}, E^\bullet_{\pm}, R^\bullet_{\pm}, E^\bullet$ be as in the Propositions 2.2.1, 2.2.9 and 2.2.10, where $\bullet = -, H, \eta$ with "$-$" symbolizing no index. Furthermore, let $S(z)$ as in Definition 1.2.2. Then we have the following estimates*

1. *for $\bullet = -, H$ and for $z \in \Omega_i \subset \Omega$*

$$\begin{aligned} &\|\partial^\beta_{z\overline{z}} R^\bullet_{\pm}\|, \|\partial^\beta_{z\overline{z}} E^\bullet_{\pm}\| = \mathcal{O}\left(h^{-|\beta|}\right), \\ &|\partial^\beta_{z\overline{z}} E^H_{-+}| = \mathcal{O}\left(h^{-(|\beta|-\frac{1}{2})} \mathrm{e}^{-\frac{S(z)}{h}}\right), \quad \|\partial^\beta_{z\overline{z}} E^\bullet\| = \mathcal{O}\left(h^{-(|\beta|+1/2)}\right). \end{aligned}$$

2. *for $\bullet = -, \eta$ and for $z \in \Omega^{a,b}_\eta \subset \Omega$*

$$\begin{aligned} &\|\partial^\beta_{z\overline{z}} R^\bullet_{\pm}\|, \|\partial^\beta_{z\overline{z}} E^\bullet_{\pm}\| = \mathcal{O}\left(\eta^{\frac{|\beta|}{2}} h^{-|\beta|}\right), \\ &|\partial^\beta_{z\overline{z}} E^\eta_{-+}| = \mathcal{O}\left(\eta^{\frac{|\beta|+1/2}{2}} h^{-(|\beta|-\frac{1}{2})} \mathrm{e}^{-\frac{\asymp\eta^{3/2}}{h}}\right), \\ &\|\partial^\beta_{z\overline{z}} E^\bullet\| = \mathcal{O}\left(\eta^{\frac{|\beta|-1/2}{2}} h^{-(|\beta|+1/2)}\right). \end{aligned}$$

Proof. Recall the definition of $R_\pm$ and $E_\pm$ given in Proposition 2.2.1. By the estimates on the z- and $\overline{z}$-derivatives of e_0 and f_0 given in Proposition 2.1.11, we may conclude for $z \in \Omega$ that

$$\begin{aligned} &\|\partial^\beta_{z\overline{z}} E_+\|_{\mathbb{C}\to L^2}, \|\partial^\beta_{z\overline{z}} R_+\|_{H^1\to\mathbb{C}} \leq \|\partial^\beta_{z\overline{z}} e_0\|_{L^2} = \mathcal{O}\left(\eta^{\frac{|\beta|}{2}} h^{-|\beta|}\right), \\ &\|\partial^\beta_{z\overline{z}} E_-\|_{L^2\to\mathbb{C}}, \|\partial^\beta_{z\overline{z}} R_-\|_{\mathbb{C}\to L^2} \leq \|\partial^\beta_{z\overline{z}} f_0\|_{L^2} = \mathcal{O}\left(\eta^{\frac{|\beta|}{2}} h^{-|\beta|}\right), \end{aligned} \tag{2.2.27}$$

and thus prove the corresponding "-"-cases in the Proposition. The estimates for the other cases of $R^\bullet_\pm$ and $E^\bullet_\pm$ then follow from (2.2.27), (2.2.28) and (2.2.32).

Recall from Proposition 2.2.1 that $\mathscr{E}(z)\mathscr{P}(z) = 1$. Thus, note that

$$\begin{aligned} &\partial_z \mathscr{E}(z) + \mathscr{E}(z)(\partial_z \mathscr{P}(z))\mathscr{E}(z) = 0, \\ &\partial_{\overline{z}} \mathscr{E}(z) + \mathscr{E}(z)(\partial_{\overline{z}} \mathscr{P}(z))\mathscr{E}(z) = 0, \end{aligned}$$

which implies

$$\begin{aligned} \partial_z E &= -E(\partial_z (P_h - z))E - E_+(\partial_z R_+)E - E(\partial_z R_-)E_- \\ &= E^2 - E_+(\partial_z R_+)E - E(\partial_z R_-)E_- \end{aligned}$$

and

$$\partial_{\overline{z}} E(z) = -E_+(z)(\partial_{\overline{z}} R_+)E(z) - E(z)(\partial_{\overline{z}} R_-)E_-(z).$$

Thus, by induction we conclude from this, from (2.2.27) and from Proposition 2.2.1 that for $z \in \Omega$

$$\|\partial^\beta_{z\overline{z}} E(z)\| = \mathcal{O}\left(\eta^{\frac{|\beta|-1/2}{2}} h^{-(|\beta|+\frac{1}{2})}\right).$$

The estimates on $\|\partial^\beta_{z\overline{z}} E^\bullet(z)\|$, for $\bullet = \eta, H$, can be conclude by following the same steps and by using the corresponding estimates on $R^\bullet_\pm$ and $E^\bullet_\pm$ and the Propositions 2.2.9 and 2.2.10.

It remains to prove the estimates on $|\partial^\beta_{z\overline{z}} E^\eta_{-+}(z)|$ and $|\partial^\beta_{z\overline{z}} E^H_{-+}(z)|$: let us first consider the case where $z \in \Omega_i \subset \Omega$. Recall (2.2.20) and recall from Proposition 2.2.9 that the prefactor $V^H(z)$ has bounded z- and $\overline{z}$-derivatives of order $\mathcal{O}(\sqrt{h})$. Thus, the statement follows immediately.

In the case where $z \in \Omega^{a,b}_\eta \subset \Omega$, recall (2.2.20) and from Proposition 2.2.10 that the prefactor $V^\eta(z)$ has bounded z- and $\overline{z}$-derivatives of order $\mathcal{O}(\sqrt{h\sqrt{\eta}})$. Using that

$$\begin{aligned} &\mathrm{e}^{\frac{i}{h}\int_{x_+}^{x_- - 2\pi}(z-g(y))dy} - \mathrm{e}^{\frac{i}{h}\int_{x_+}^{x_-}(z-g(y))dy} \\ &\qquad = \mathrm{e}^{\frac{i}{\tilde{h}}\int_{\tilde{x}_+}^{\tilde{x}_- - 2\pi/\sqrt{\eta}}(\tilde{z}-\tilde{g}(\tilde{y}))d\tilde{y}} - \mathrm{e}^{\frac{i}{\tilde{h}}\int_{\tilde{x}_+}^{\tilde{x}_-}(\tilde{z}-\tilde{g}(\tilde{y}))d\tilde{y}}, \end{aligned}$$

(2.1.5) implies

$$|\partial^\beta_{z\overline{z}} E^\eta_{-+}(z)| = \eta^{-|\beta|}|\partial^\beta_{\tilde{z}\overline{\tilde{z}}} E^\eta_{-+}(z)| = \mathcal{O}\left(\eta^{\frac{|\beta|+1/2}{2}} h^{-(|\beta|-\frac{1}{2})}\mathrm{e}^{\frac{\asymp\eta^{3/2}}{h}}\right). \qquad \square$$

Proof of Proposition 2.2.12. Let $\bullet = H, \eta$ denote the quasimodes and elements of the Grushin problems corresponding to the different zones of z.

Since $\mathcal{P}^\bullet\mathcal{E}^\bullet : L^2(S^1)\times\mathbb{C} \longrightarrow L^2(S^1)\times\mathbb{C}$ let us introduce the following norm for an operator-valued matrix $A: L^2(S^1)\times\mathbb{C} \longrightarrow L^2(S^1)\times\mathbb{C}$:

$$\|A\|_\infty := \max_{1\le i\le 2}\sum_{j=1}^{2}\|A_{ij}\|,$$

where $\|A_{ij}\|$ denotes the respective operator norm for A_{ij}. Next, note that

$$\mathcal{P}\mathcal{E}^\bullet = \left(\mathcal{P}^\bullet + (\mathcal{P}-\mathcal{P}^\bullet)\right)\mathcal{E}^\bullet = 1 + (\mathcal{P}-\mathcal{P}^\bullet)\mathcal{E}^\bullet.$$

Estimates for $(\mathcal{P}-\mathcal{P}^\bullet)$ Recall the definition of $\mathcal{P}$ and of $\mathcal{P}^\bullet$ from the Propositions 2.2.1, 2.2.9 and 2.2.10 and note that

$$\mathcal{P}-\mathcal{P}^\bullet = \begin{pmatrix} 0 & R_- - R_-^\bullet \\ R_+ - R_+^\bullet & 0 \end{pmatrix}.$$

We will now prove that for all $(n,m)\in\mathbb{N}^2$

$$\begin{aligned} \|\partial^\beta_{z\overline{z}}(R_+ - R_+^\bullet)\|_{H^1(S^1)\to\mathbb{C}} &\le \|\partial^\beta_{z\overline{z}}(e_0 - \chi_+^\bullet e_+^\bullet)\| \\ &= \begin{cases} \mathcal{O}\left(h^{-|\beta|}\mathrm{e}^{-\frac{1}{Ch}}\right), \text{ for } z\in\Omega,\ \mathrm{dist}(\Omega,\partial\Sigma) > 1/C, \\ \mathcal{O}\left(\eta^{\frac{|\beta|}{2}} h^{-|\beta|}\mathrm{e}^{-\frac{\asymp\eta^{\frac{3}{2}}}{h}}\right), \text{ for } z\in\Omega^a_\eta, \end{cases} \end{aligned} \tag{2.2.28}$$

where the first estimate follows from the Cauchy-Schwartz inequality. Note that

$$\|\partial^\beta_{z\overline{z}}(e_0 - \chi_+^\bullet e_+^\bullet)\| \le \|\partial^\beta_{z\overline{z}}(e^\bullet_{wkb} - \chi_+^\bullet e_+^\bullet)\| + \|\partial^\beta_{z\overline{z}}(e_0 - e^\bullet_{wkb})\|. \tag{2.2.29}$$

By Proposition 2.1.11 it remains to prove the desired estimate on $\|\partial^\beta_{z\overline{z}}(e^\bullet_{wkb} - \chi_+^\bullet e_+^\bullet)\|$. Recall the definition of the quasimodes $e^\bullet_{wkb}$ and $e^\bullet_+$ from Section 2.1 and from Proposition 2.2.8.

Let us first consider the case of $z\in\Omega$ with $\mathrm{dist}(\Omega,\partial\Sigma) > 1/C$: recall from Proposition 2.2.9 that all z- and $\overline{z}$-derivatives of χ_+ are bounded independently of $h>0$, whereas for the derivatives of χ_e we have (2.1.16). Thus

$$\partial^\beta_{z\overline{z}}\chi_+, \partial^\beta_{z\overline{z}}\chi_e = \mathcal{O}(h^{-|\beta|/2}).$$

Thus, since $\chi_e(\cdot,z) \succ \chi_+$ for all $z\in\overline{\Omega}_i$, which implies that $x_+(z)\notin \operatorname{supp}(\chi_e(\cdot,z)-\chi_+)$ for all $z\in\overline{\Omega}_i$, the Leibniz rule then implies

$$\left\| \partial^{\beta}_{z\bar{z}}\left((\chi_e(\cdot,z)-\chi_+)\,\mathrm{e}^{\frac{i}{h}\phi_+(\cdot,z)}\right)\right\| \leq \mathcal{O}\left(h^{-|\beta|}\mathrm{e}^{-\frac{F}{h}}\right). \tag{2.2.30}$$

where $F>0$ is given by the infimum of $\operatorname{Im}\phi(x,z)$ over all $z\in\overline{\Omega}$ and all

$$x\in\left(\bigcup_{z\in\overline{\Omega}}\operatorname{supp}(\chi_e(\cdot,z))\right)\backslash\{x\in I_+:\ \chi_+\equiv 1\}.$$

Note that $F>0$ is strictly positive because $x_-(z)\notin\overline{I_+}$ for all $z\in\overline{\Omega}$ and $\chi_+\in\mathcal{C}_0^\infty(I_+)$ (cf. Propositions 2.2.9 and 2.2.8).

Recall that $h^{-1/4}a(z;h)$ and $c_+(z;h)$ are the normalization factors of e_{wkb} and e_+ (cf. (2.1.7) and Proposition 2.2.8). Hence, for $z\in\Omega_i$,

$$h^{-\frac{1}{4}}\partial^{\beta}_{z\bar{z}}a(z;h),\ \ \partial^{\beta}_{z\bar{z}}c_+(z;h)=\mathcal{O}\left(h^{-(|\beta|+1/2)}\right).$$

Thus the Leibniz rule and (2.2.30) imply

$$\begin{aligned}
&|\partial^{\beta}_{z\bar{z}}c_+(z;h)-\partial^{\beta}_{z\bar{z}}h^{-1/4}a(z;h)|\\
&\quad=\left|\partial^{\beta}_{z\bar{z}}\frac{\left\|\left(\chi_e(\cdot,z)\mathrm{e}^{\frac{i}{h}\phi_+(\cdot,z)}\right)\right\|-\left\|\left(\chi_+\mathrm{e}^{\frac{i}{h}\phi_+(\cdot,z)}\right)\right\|}{\left\|\left(\chi_e(\cdot,z)\mathrm{e}^{\frac{i}{h}\phi_+(\cdot,z)}\right)\right\|\left\|\left(\chi_+\mathrm{e}^{\frac{i}{h}\phi_+(\cdot,z)}\right)\right\|}\right|\\
&\quad=\mathcal{O}\left(h^{-(|\beta|+1/2)}\mathrm{e}^{-\frac{F}{h}}\right).
\end{aligned}$$

Since $h^{-\frac{1}{4}}a(z;h), c_+(z;h)=\mathcal{O}(h^{-\frac{1}{4}})$, the Leibniz rule and the above imply that for $z\in\Omega_i$

$$\|\partial^{\beta}_{z\bar{z}}(e_{wkb}-\chi_+e_+)\|\leq\mathcal{O}\left(h^{-(|\beta|+1/2)}\mathrm{e}^{-\frac{F}{h}}\right).$$

Thus there exists a constant $C>0$, for $h>0$ small enough, such that for $z\in\Omega_i$

$$\|\partial^{\beta}_{z\bar{z}}(e_{wkb}-\chi_+e_+)\|=\mathcal{O}\left(h^{-|\beta|}\mathrm{e}^{-\frac{1}{Ch}}\right). \tag{2.2.31}$$

Now let us consider the case $z\in\Omega\cap\Omega_\eta^{a,b}$: recall the quasimodes e^{η}_{wkb} and e^{η}_+ as given in Definition 2.1.6 and Proposition 2.2.8. A rescaling argument similar to the one in the proof of Proposition 2.1.11 then implies

$$\|\partial^{\beta}_{z\bar{z}}(e^{\eta}_{wkb}-\chi^{\eta}_+e^{\eta}_+)\|=\mathcal{O}\left(\eta^{\frac{|\beta|+3/2}{2}}h^{-(|\beta|+1/2)}\mathrm{e}^{-\frac{\asymp\eta^{\frac{3}{2}}}{h}}\right).$$

Absorbing the factor $\eta^{3/4}h^{-1/2}$ into $\mathrm{e}^{-\frac{\asymp\eta^{\frac{3}{2}}}{h}}$ then yields the desired estimate.

It is possible to achieve an analogous estimate for $R_- - R_-^{\bullet}$, namely that for all $z\in\Omega$ and for all $(n,m)\in\mathbb{N}^2$

$$\begin{aligned}
&\|\partial^{\beta}_{z\bar{z}}(R_--R_-^{\bullet})\|_{\mathbb{C}\to H^1(S^1)}=\|\partial^{\beta}_{z\bar{z}}(f_0-\chi_-^{\bullet}e_-^{\bullet})\|\\
&=\begin{cases}\mathcal{O}\left(h^{-|\beta|}\mathrm{e}^{-\frac{1}{Ch}}\right), \text{ for } z\in\Omega,\ \operatorname{dist}(\Omega,\partial\Sigma)>1/C,\\
\mathcal{O}\left(\eta^{\frac{|\beta|}{2}}h^{-|\beta|}\mathrm{e}^{-\frac{\asymp\eta^{\frac{3}{2}}}{h}}\right), \text{ for } z\in\Omega^{a}_{\eta},\end{cases}
\end{aligned} \tag{2.2.32}$$

This can be achieved by analogous reasoning as for the estimate on $R_+ - R_+^{\bullet}$.

A formula for E_{-+} Using (2.2.28), (2.2.32), it follows that for $h>0$ small enough

$$\|(\mathcal{P}-\mathcal{P}^{\bullet})\mathcal{E}^{\bullet}\|_\infty\ll 1.$$

Thus, $1+(\mathscr{P}-\mathscr{P}^\bullet)\mathscr{E}^\bullet$ is invertible by the Neumann series, wherefore

$$\mathscr{P}\mathscr{E}^\bullet\left[1+(\mathscr{P}-\mathscr{P}^\bullet)\mathscr{E}^\bullet\right]^{-1}=1.$$

We conclude that

$$\mathscr{E}=\mathscr{E}^\bullet\sum_{n\geq 0}(-1)^n\left[(\mathscr{P}-\mathscr{P}^\bullet)\mathscr{E}^\bullet\right]^n.$$

Define $g_- := R_- - R_-^\bullet$ and $g_+ := R_+ - R_+^\bullet$. Hence, by Propositions 2.2.9 and 2.2.10 as well as by (2.2.29) and (2.2.28), there exists a constant $C>0$ such that

$$(\mathscr{P}-\mathscr{P}^\bullet)\mathscr{E}^\bullet=\begin{pmatrix} g_-E_-^\bullet & g_-E_{-+}^\bullet \\ g_+E^\bullet & g_+E_+^\bullet\end{pmatrix}=\begin{pmatrix}\mathscr{O}\left(\mathrm{e}^{-\frac{1}{Ch}}\right) & E_{-+}^\bullet\mathscr{O}\left(\mathrm{e}^{-\frac{1}{Ch}}\right)\\ \mathscr{O}\left(\mathrm{e}^{-\frac{1}{Ch}}\right) & \mathscr{O}\left(\mathrm{e}^{-\frac{1}{Ch}}\right)\end{pmatrix}.$$

By induction it follows that for $n\in\mathbb{N}$

$$\left[(\mathscr{P}-\mathscr{P}^\bullet)\mathscr{E}^\bullet\right]^n=\begin{pmatrix}\mathscr{O}\left(\mathrm{e}^{-\frac{n}{Ch}}\right) & E_{-+}^\bullet\mathscr{O}\left(\mathrm{e}^{-\frac{n}{Ch}}\right)\\ \mathscr{O}\left(\mathrm{e}^{-\frac{n}{Ch}}\right) & \mathscr{O}\left(\mathrm{e}^{-\frac{n}{Ch}}\right)\end{pmatrix}.$$

We conclude that

$$E_{-+}(z)=E_{-+}^\bullet\left(1+\sum_{n\geq 1}\mathscr{O}\left(\mathrm{e}^{-\frac{n}{Ch}}\right)\right)=E_{-+}^\bullet\left(1+\mathscr{O}\left(\mathrm{e}^{-\frac{1}{Ch}}\right)\right).$$

Finally, by the estimates on g_+ and g_+ obtained above and by the estimates given in Proposition 2.2.13 we conclude the desired estimates on the z- and $\overline{z}$-derivatives of the error term. ☐

2.3 | Grushin problem for the perturbed operator P_h^δ

For $\delta>0$ small enough, we can use the Grushin problem for the unperturbed operator P_h to gain a well-posed Grushin problem for the perturbed operator P_h^δ.

Proposition 2.3.1 ([67]). *Let $z\in\Omega\Subset\Sigma$, let $h^{2/3}\ll\eta\leq$ const. and let R_-,R_+ be as in Proposition 2.2.1. Then*

$$\mathscr{P}_\delta(z):=\begin{pmatrix}P_h^\delta-z & R_-\\ R_+ & 0\end{pmatrix}:\ H^1(S^1)\times\mathbb{C}\longrightarrow L^2(S^1)\times\mathbb{C}$$

is bijective with the bounded inverse

$$\mathscr{E}_\delta(z)=\begin{pmatrix}E^\delta(z) & E_+^\delta(z)\\ E_-^\delta(z) & E_{-+}^\delta(z)\end{pmatrix}$$

where

$$\begin{aligned}
E^\delta(z)&=E(z)+\mathscr{O}_{\eta^{-1/2}}\left(\delta h^{-2}\right)=\mathscr{O}(\eta^{-1/4}h^{-1/2})\\
E_-^\delta(z)&=E_-(z)+\mathscr{O}\left(\delta\eta^{-1/4}h^{-3/2}\right)=\mathscr{O}(1)\\
E_+^\delta(z)&=E_+(z)+\mathscr{O}\left(\delta\eta^{-1/4}h^{-3/2}\right)=\mathscr{O}(1)
\end{aligned}$$

and

$$\begin{aligned}
E_{-+}^\delta(z)&=E_{-+}(z)-\delta\left(E_-Q_\omega E_+ +\sum_{n=1}^{\infty}(-\delta)^nE_-Q_\omega(EQ_\omega)^nE_+\right)\\
&=E_{-+}(z)-\delta\left(E_-Q_\omega E_+ +\mathscr{O}(\delta\eta^{-1/4}h^{-5/2})\right)
\end{aligned}\tag{2.3.1}$$

Proof. The statement follows immediately from Proposition 2.2.1 by use of the Neumann series. ☐

By (2.2.2) we get

$$E_-Q_\omega E_+ = \sum_{|j|,|k|\leq\left\lfloor\frac{C_1}{h}\right\rfloor} \alpha_{j,k}(e_0|e^k)\cdot(e^j|f_0) = \sum_{|j|,|k|\leq\left\lfloor\frac{C_1}{h}\right\rfloor} \alpha_{j,k}\,\widehat{e}_0(k)\overline{\widehat{f}_0(j)}.$$

Recall from Corollary 1.1.5 that the random variables satisfy $\alpha\in B(0,C/h)$. For a more convenient notation we make the following definition:

Definition 2.3.2. For $x\in\mathbb{R}$ we shall denote the Gauss brackets by $\lfloor x\rfloor := \max\{k\in\mathbb{Z}:\ k\leq x\}$. Let $C_1>0$ be big enough as above and define $N:=(2\lfloor\frac{C_1}{h}\rfloor+1)^2$. For $z\in\Omega\Subset\Sigma$, let $X(z)=(X_{j,k}(z))_{|j|,|k|\leq\lfloor\frac{C_1}{h}\rfloor}\in\mathbb{C}^N$ be given by

$$X_{j,k}(z)=\widehat{e}_0(z;k)\overline{\widehat{f}_0(z;j)},\quad \text{for } |j|,|k|\leq\left\lfloor\frac{C_1}{h}\right\rfloor.$$

Thus, for $z\in\Omega\Subset\Sigma$ and $\alpha\in B(0,C/h)\subset\mathbb{C}^N$

$$E^\delta_{-+}(z)=E_{-+}(z)-\delta\left[X(z)\cdot\alpha+T(z,\alpha)\right], \tag{2.3.2}$$

where the dot-product $X(z)\cdot\alpha$ is the bilinear one, and

$$T(z,\alpha):=\sum_{n=1}^{\infty}(-\delta)^n E_-Q_\omega(EQ_\omega)^n E_+ = \mathcal{O}(\delta\eta^{-1/4}h^{-5/2}), \tag{2.3.3}$$

where the estimate comes from Proposition 2.3.1. Note that $T(z,\alpha)$ is $\mathscr{C}^\infty$ in z and holomorphic in α in a ball of radius C/h, $B(0,C/h)\subset\mathbb{C}^N$, by Corollary 1.1.5.

Proposition 2.3.3. *Let $z\in\Omega\Subset\Sigma$, let $X(z)$ be as in Definition 2.3.2. Let $h|k|\geq C$ for $C>0$ large enough, then the Fourier coefficients satisfy*

$$\widehat{e}_0(z;k),\ \widehat{f}_0(z;k)=\mathcal{O}\left(|k|^{-M}\mathrm{dist}\,(\Omega,\partial\Sigma)^{-\frac{M}{2}}\right),\quad \mathrm{dist}\,(\Omega,\partial\Sigma)\gg h^{\frac{2}{3}}$$

for all $M\in\mathbb{N}$. In particular

$$\|X(z)\|_2=1+\mathcal{O}\left(h^\infty\right).$$

Proof. We will show the proof in the case of $e_0(z)$ since the case of $f_0(z)$ is similar. Let us first suppose that $z\in\Omega$ with $\mathrm{dist}\,(\Omega,\partial\Sigma)>1/C$. Recall the definition of the quasimode e_{wkb} given in (2.1.7). By Proposition 2.2.7

$$\widehat{e}_0(z;k)=\int\left(e_{wkb}(z,x)+\mathcal{O}_{C^\infty}\left(\mathrm{e}^{-\frac{S}{2h}}\right)\right)\mathrm{e}^{-ikx}dx.$$

For $k\in\mathbb{Z}\backslash\{0\}$, repeated integration by parts using the operator

$${}^tL:=\frac{i}{k}\frac{d}{dx}$$

applied to the error term yields by Proposition 2.2.7 that for all $n\in\mathbb{N}$

$$\widehat{e}_0(z;k)=\int e_{wkb}(z,x)\mathrm{e}^{-ikx}dx+\mathcal{O}\left(|k|^{-n}h^\infty\right).$$

Define the phase function $\Phi(x,z):=(\phi_+(x,z)h^{-1}-kx)$. Since $h|k|\geq C$ is large enough and since Ω is relatively compact, it follows that

$$|\partial_x\Phi(x,z)|=|\partial_x\phi_+(x,z)h^{-1}-k|\geq C_1|k|>0.$$

Repeated integration by parts using the operator

$${}^tL':=\frac{1}{\partial_x\Phi(x,z)}D_x$$

yields that for all $n \in \mathbb{N}$

$$\int e_{wkb}(z,x)\mathrm{e}^{-ikx}dx = \mathcal{O}\left(|k|^{-n}\right).$$

Thus, for all $n \in \mathbb{N}$

$$\widehat{e_0}(z;k) = \mathcal{O}\left(|k|^{-n}\right).$$

For $z \in \Omega \cap \Omega_\eta^a$ one performs a similar rescaling argument as in the proof of Proposition 2.1.11. Since in the rescaled coordinates $\tilde{k} = \sqrt{\eta}k$, we conclude that for all $n \in \mathbb{N}$

$$|\widehat{e_0}(z;k)| \leq \mathcal{O}\left(\eta^{-\frac{n}{2}}|k|^{-n}\right).$$

Finally, by definition 2.3.2, Parseval identity and the estimates on the Fourier coefficients above, it follows that

$$\|X(z)\|^2 = \sum_{|j|,|k|\leq\lfloor\frac{C_1}{h}\rfloor} |\widehat{e_0}(z;j)|^2|\widehat{f_0}(z;k)|^2 = (e_0(z)|e_0(z))(f_0(z)|f_0(z)) + \mathcal{O}\left(h^\infty\right).$$

Since $\|e_0\|, \|f_0\| = 1$, we conclude the second statement of the Proposition. □

The following is an extension of Proposition 2.3.3.

Proposition 2.3.4. *Let $z \in \Omega \Subset \Sigma$, let $X(z)$ be as in Definition 2.3.2. Let $h|k| \geq C$ for $C > 0$ large enough, then for* $\mathrm{dist}(\Omega, \partial\Sigma) \gg h^{\frac{2}{3}}$ *and for all $n, m \in \mathbb{N}_0$*

$$\partial_z^n\partial_{\overline{z}}^m\widehat{e_0}(z;k), \partial_z^n\partial_{\overline{z}}^m\widehat{f_0}(z;k) = \left(|k|^{-M}\mathrm{dist}(\Omega,\partial\Sigma)^{-\frac{M}{2}}\right).$$

Furthermore,

$$\|\partial_z^n\partial_{\overline{z}}^m X(z)\| = \mathcal{O}\left(\mathrm{dist}(\Omega,\partial\Sigma)^{\frac{n+m}{2}}h^{-(n+m)}\right).$$

Proof. Since

$$\partial_z^n\partial_{\overline{z}}^m\widehat{e_0}(z;k) = \int \partial_z^n\partial_{\overline{z}}^m e_0(z,x)\mathrm{e}^{-ikx}dx.$$

We then conclude similar to the proof of Proposition 2.3.3 that for all $N \in \mathbb{N}$

$$|\partial_z^n\partial_{\overline{z}}^m\widehat{e_0}(z;k)| = \mathcal{O}\left(\eta^{-\frac{N}{2}}|k|^{-N}\right).$$

The second statement of the Proposition is a direct consequence of Parseval's identity and Proposition 2.1.11. □

2.4 | Connections with symplectic volume and tunneling effects

The first two terms of the effective Hamiltonian E_{-+}^δ for the perturbed operator P_h^δ (cf. (2.3.2)) have a relation to the symplectic volume form on T^*S^1 and to the tunneling effects described in Section 2.2.2.

2.4.1 – Link with the symplectic volume

Proposition 2.4.1. *Let $z \in \Omega \Subset \Sigma$ and let p be as in (1.1.7) and $\rho_\pm$ be as in (1.1.14). Let $X(z)$ be as in Definition 2.3.2. Then we have for $h>0$ small enough and $h^{2/3} \ll \eta \leq \mathrm{const}$.*

$$(\partial_z X|\partial_z X) - \frac{|(\partial_z X|X)|^2}{\|X\|^2} = \frac{1}{h}\left(\frac{i}{\{p,\overline{p}\}(\rho_+(z))} - \frac{i}{\{p,\overline{p}\}(\rho_-(z))}\right) + \mathcal{O}(\eta^{-2}),$$

where

$$|\{p,\overline{p}\}(\rho_\pm)| \asymp \sqrt{\eta}.$$

The $\partial^\beta_{z\overline{z}}$ derivatives of the error term $\mathcal{O}(\eta^{-2})$ are of order $\mathcal{O}\left(\eta^{\frac{|\beta|}{2}-2}h^{-\frac{|\beta|}{2}}\right)$.

Proposition 2.4.2. *Let $z \in \Omega \Subset \Sigma$, let p be as in (1.1.7), let $\rho_\pm$ be as in (1.1.14), and let $d\xi \wedge dx$ be the symplectic form on T^*S^1. Then,*

$$\begin{aligned}\frac{1}{h}\left(\frac{i}{\{p,\overline{p}\}(\rho_+(z))} - \frac{i}{\{p,\overline{p}\}(\rho_-(z))}\right)L(dz) &= \frac{1}{2h}(d\xi_- \wedge dx_- - d\xi_+ \wedge dx_+)\\ &= \frac{1}{2h}p_*(d\xi\wedge dx)\end{aligned}$$

Proof of Proposition 2.4.2. In the following we will conform to ideas from [32, 4, 67]: Since $p(x_\pm,\xi_\pm) = z$, we find the following system of linear equations

$$\begin{cases} p'_x \cdot \partial_z x_\pm + p'_\xi \cdot \partial_z \xi_\pm = 1\\ p'_x \cdot \partial_{\overline{z}} x_\pm + p'_\xi \cdot \partial_{\overline{z}} \xi_\pm = 0\end{cases}$$

and since $x_\pm, \xi_\pm \in \mathbb{R}$

$$\begin{cases} p'_x \cdot \partial_z x_\pm + p'_\xi \cdot \partial_z \xi_\pm = 1\\ \overline{p}'_x \cdot \partial_z x_\pm + \overline{p}'_\xi \cdot \partial_z \xi_\pm = 0.\end{cases}$$

This system can be solved and we find

$$\partial_z x_\pm = \frac{-\overline{p}'_\xi}{\{p,\overline{p}\}}(\rho_\pm),\ \partial_{\overline{z}} x_\pm = \frac{p'_\xi}{\{p,\overline{p}\}}(\rho_\pm) \tag{2.4.1}$$

and

$$\partial_z \xi_\pm = \frac{\overline{p}'_x}{\{p,\overline{p}\}}(\rho_\pm),\ \partial_{\overline{z}} \xi_\pm = \frac{-p'_x}{\{p,\overline{p}\}}(\rho_\pm).$$

Hence we have

$$d\xi_\pm \wedge dx_\pm = \left(\partial_z\xi_\pm\partial_{\overline{z}}x_\pm - \partial_{\overline{z}}\xi_\pm\partial_z x_\pm\right)dz\wedge d\overline{z} = \left(\frac{1}{\{p,\overline{p}\}}(\rho_\pm)\right)dz\wedge d\overline{z}.$$

Since the Lebesgue measure with the standard orientation of $\mathbb{C}$ can be represented as

$$L(dz) \simeq \frac{i}{2}dz\wedge d\overline{z},$$

and the statement of the Proposition follows. ☐

To prove Proposition 2.4.1 we first prove the following result.

Lemma 2.4.3. *Let $z \in \Omega \Subset \Sigma$ such that* $\mathrm{dist}(\Omega, \partial\Sigma) > 1/C$ *and let $g \in \mathscr{C}^\infty(\mathbb{C})$ and $\rho_\pm$ be as in (1.1.14). Let e_{wkb} and f_{wkb} be as in (2.1.7) and (2.1.8). Let $\Pi_{e_{wkb}}: L^2(S^1) \to L^2(S^1)$ and $\Pi_{f_{wkb}}: L^2(S^1) \to L^2(S^1)$ denote the orthogonal projections onto the subspaces spanned by e_{wkb} and f_{wkb} respectively. Then,*

$$\|(1-\Pi_{e_{wkb}})\partial_z e_{wkb}(\cdot,z)\|^2 = \frac{-1}{2h\mathrm{Im}\, g'(x_+(z))} + \mathcal{O}(1),$$

$$\|(1-\Pi_{f_{wkb}})\partial_{\overline{z}} f_{wkb}(\cdot,z)\|^2 = \frac{1}{2h\mathrm{Im}\, g'(x_-(z))} + \mathcal{O}(1).$$

Remark 2.4.4. In the following, we shall regard z as a fixed parameter. Hence, by the support of functions depending on both x and z we mean the support with respect to the variable x.

Proof. We will consider only the case of e_{wkb} since the case of f_{wkb} is similar. One calculates

$$\partial_z e_{wkb}(x,z) = h^{-\frac{1}{4}}\Big\{ \partial_z\chi_e(x,z)a(z;h) + \chi_e(x,z)\partial_z a(z;h) + \chi_e(x,z)a(z;h)\frac{i}{h}\partial_z\phi_+(x,z)\Big\}\mathrm{e}^{\frac{i}{h}\phi_+(x,z)}. \tag{2.4.2}$$

Thus

$$(\partial_z e_{wkb}|e_{wkb}) = h^{-\frac{1}{2}}\int\Big(\big(\partial_z\chi_e(x,z)\big)|a(z;h)|^2 + (\partial_z a(z;h))\,\overline{a(z;h)}\chi_e(x,z) + |a(z;h)|^2\chi_e(x,z)\frac{i}{h}\partial_z\phi_+(x,z)\Big)\chi_e(x,z)\mathrm{e}^{-\frac{\Phi(x,z)}{h}}dx, \tag{2.4.3}$$

where

$$\Phi(x,z) := -i(\phi_+(x,z)-\overline{\phi_+(x,z)}) = 2\mathrm{Im}\int_{x_+(z)}^{x}(z-g(y))dy. \tag{2.4.4}$$

First, we will compute

$$h^{-\frac{1}{2}}\int\big(\partial_z\chi_e(x,z)\big)\chi_e(x,z)|a(z;h)|^2\mathrm{e}^{-\frac{\Phi(x,z)}{h}}dx. \tag{2.4.5}$$

Using (2.1.16) and the fact that $\partial_z\chi_e(z,\cdot)$ has support in $]x_- - 2\pi, x_- - 2\pi + h^{1/2}[\cup]x_- - h^{1/2}, x_-[$, Taylor expansion of $\Phi(\cdot,z)$ at x_- and $x_- - 2\pi$ yields that

$$\mathrm{e}^{-\frac{\Phi(x,z)}{h}} \leq \mathcal{O}\Big(\mathrm{e}^{-\frac{2S}{h}}\Big),$$

uniformly in $]x_- - 2\pi, x_- - 2\pi + h^{1/2}[\cup]x_- - h^{1/2}, x_-[$. Here S is as in Definition 1.2.2. Now, applying this and (2.1.16) to (2.4.5), yields

$$h^{-\frac{1}{2}}|a(z;h)|^2\int\partial_z\chi_e(x,z)\chi_e(x,z)\mathrm{e}^{-\frac{\Phi(x,z)}{h}}dx = \mathcal{O}\Big(h^{-\frac{1}{2}}\mathrm{e}^{-\frac{2S}{h}}\Big). \tag{2.4.6}$$

Next, we will treat the other two contributions to (2.4.3). First, consider

$$h^{-\frac{1}{2}}(\partial_z a(z;h))\,\overline{a(z;h)}\int\chi_e(x,z)^2\mathrm{e}^{-\frac{\Phi(x,z)}{h}}dx.$$

Since $h^{-\frac{1}{2}}|a(z;h)|^2$ is the normalization factor of $\|e_{wkb}\|^2$ we see that

$$h^{-\frac{1}{2}}\partial_z a(z;h)\overline{a(z;h)}\int\chi_e(x,z)^2\mathrm{e}^{-\frac{\Phi(x,z)}{h}}dx = \frac{\partial_z a(z;h)}{a(z;h)}. \tag{2.4.7}$$

Let us now turn to the third contribution to (2.4.3)

$$I_h := h^{-\frac{1}{2}}|a(z;h)|^2\int\frac{i}{h}\partial_z\phi_+(x,z)\chi_e(x,z)^2\mathrm{e}^{-\frac{\Phi(x,z)}{h}}dx.$$

The stationary phase method implies together with (2.1.10) that

$$I_h = \frac{i}{h}\partial_z\phi_+(x_+(z),z) + \mathcal{O}(1). \tag{2.4.8}$$

Thus, by combining (2.4.6), (2.4.7) and (2.4.8)

$$(\partial_z e_{wkb}|e_{wkb}) = \frac{i}{h}\partial_z\phi_+(x_+(z),z) + \mathcal{O}(1)$$

and thus

$$(\partial_z e_{wkb}|e_{wkb})e_{wkb}(x,z) = h^{-\frac{1}{4}}\left\{a(z;h)\frac{i}{h}\partial_z\phi_+(x_+(z),z)+\mathscr{O}(1)\right\}\chi_e(x,z)\mathrm{e}^{\frac{i}{h}\phi_+(x,z)}. \tag{2.4.9}$$

Subtract (2.4.9) from (2.4.2) and note that the term $a(z;h)\partial_z\chi_e(x,z)\mathrm{e}^{\frac{i}{h}\phi_+(x,z)}$ is exponentially small in h like in (2.4.6). Thus

$$\begin{aligned}(1-\Pi_{e_{wkb}})&\partial_z e_{wkb}(x,z)\\ &=\frac{\mathrm{e}^{\frac{i}{h}\phi_+(x,z)}}{h^{1/4}}\left\{a(z;h)\chi_e(x,z)\frac{i}{h}\left(\partial_z\phi_+(x,z)-\partial_z\phi_+(x_+(z),z)\right)\right\}+\mathscr{O}_{L^2}(1).\end{aligned} \tag{2.4.10}$$

It remains to treat

$$\begin{aligned}I_h &:= \left\| a(z;h)\chi_e(x,z)\frac{i}{h^{\frac{5}{4}}}\left(\partial_z\phi_+(x,z)-\partial_z\phi_+(x_+(z),z)\right)\mathrm{e}^{\frac{i}{h}\phi_+(x,z)}\right\|^2\\ &= h^{-\frac{1}{2}}\int \chi_e(x,z)^2|a(z;h)|^2\left|\frac{i}{h}\left(\partial_z\phi_+(x,z)-\partial_z\phi_+(x_+(z),z)\right)\right|^2\mathrm{e}^{-\frac{\Phi(x,z)}{h}}dx,\end{aligned} \tag{2.4.11}$$

where $\Phi(x,z)$ is given in (2.4.4). This can be done by the stationary phase method, as in the proof of Lemma 2.1.3. Thus

$$I_h = \sqrt{2\pi}\sum_{n=0}^{N}\frac{1}{n!}\left(\frac{h}{2}\right)^n(\Delta_y^n u)(0)+\mathscr{O}(h^{N+1}),$$

where

$$u(y) = \chi_e(\kappa^{-1}(y),z)^2\frac{|a(z;h)|^2}{|\kappa'(\kappa^{-1}(y))|}\left|\frac{i}{h}\left(\partial_z\phi_+(\kappa^{-1}(y),z)-\partial_z\phi_+(x_+(z),z)\right)\right|^2$$

and $\kappa: V\to U$ is a local $\mathscr{C}^\infty$ diffeomorphism from $V\subset\mathbb{R}$, a neighborhood of $x_+(z)$, to $U\subset\mathbb{R}$, a neighborhood of 0, such that

$$\Phi(\kappa^{-1}(x),z) = \Phi(x_+(z),z)+\frac{x^2}{2},$$

$\kappa^{-1}(0) = x_+(z)$ and

$$\frac{d\kappa}{dx}(x_+(z)) = |\partial^2_{xx}\Phi(x_+(z),z)|^{\frac{1}{2}} = \sqrt{-2\mathrm{Im}\, g'(x_+(z))}\neq 0. \tag{2.4.12}$$

This implies that $u(0)=0$ and thus we have to calculate the second order term in the above asymptotics, i.e. $\Delta_y u(y)$ is equal to

$$\begin{aligned}&\left(\Delta_y\chi_e(\kappa^{-1}(y),z)^2\frac{|a(z;h)|^2}{|\kappa'(\kappa^{-1}(y))|}\right)\left|\frac{i}{h}\left(\partial_z\phi_+(\kappa^{-1}(y),z)-\partial_z\phi_+(x_+,z)\right)\right|^2\\ &+2\frac{d}{dy}\left(\chi_e(\kappa^{-1}(y),z)^2\frac{|a(z;h)|^2}{|\kappa'(\kappa^{-1}(y))|}\right)\frac{d}{dy}\frac{1}{h^2}\left|\partial_z\phi_+(\kappa^{-1}(y),z)-\partial_z\phi_+(x_+,z)\right|^2\\ &+\chi_e(\kappa^{-1}(y),z)^2\frac{|a(z;h)|^2}{|\kappa'(\kappa^{-1}(y))|}\Delta_y\left(\left|\frac{i}{h}\left(\partial_z\phi_+(\kappa^{-1}(y),z)-\partial_z\phi_+(x_+,z)\right)\right|^2\right).\end{aligned}$$

Note that at $y=0$ the first and the second term of the right hand side vanish. By (2.1.39)

$$\Delta_y\left(\left|\frac{i}{h}\left(\partial_z\phi_+(\kappa^{-1}(y),z)-\partial_z\phi_+(x_+(z),z)\right)\right|^2\right)\Big|_{y=0} = 2h^{-2}\left|\frac{d}{dy}\kappa^{-1}(0)\right|^2.$$

Thus, since $\chi_e(\kappa^{-1}(0),z) = \chi_e(x_+(z),z)=1$ (cf. Definition 2.1.2),

$$(\Delta_y u)(0) = \frac{2|a(z;h)|^2}{h^2|\kappa'(x_+(z))|^3}.$$

Using (2.4.12)and (2.1.10), we have that

$$(\Delta_y u)(0) = \frac{1}{\sqrt{2\pi}h^2}\left(-\operatorname{Im} g'(x_+(z))\right)^{-1} + \mathscr{O}(h^{-1})$$

which yields

$$I_h = \frac{-1}{2h\operatorname{Im} g'(x_+(z))} + \mathscr{O}(1).$$

This, together with (2.4.10), yields

$$\|(1-\Pi_{e_{wkb}})\partial_z e_{wkb}(\cdot,z)\|^2 = \frac{-1}{2h\operatorname{Im} g'(x_+(z))} + \mathscr{O}(1).$$ □

Proof of Proposition 2.4.1. Recall that $e_0(z)$ (respectively $f_0(z)$) denotes an eigenfunction of the z-dependent operator $Q(z)$ (respectively $\tilde{Q}(z)$). Using Definition 2.3.2, Proposition 2.3.3, Corollary 2.3.4 and the Parseval identity one computes that

$$\begin{aligned}(\partial_z X|\partial_z X) - \frac{|(\partial_z X|X)|^2}{\|X\|^2} &= \\ &= (\partial_z e_0|\partial_z e_0) - |(\partial_z e_0|e_0)|^2 + (\partial_{\bar z} f_0|\partial_{\bar z} f_0) - |(f_0|\partial_{\bar z} f_0)|^2 + \mathscr{O}\left(h^\infty\right).\end{aligned}$$

Suppose that $z\in\Omega$ with $\operatorname{dist}(\Omega,\partial\Sigma) > 1/C$. By Corollary 2.1.13 it then follows that $(\partial_z e_0|\partial_z e_0) - |(\partial_z e_0|e_0)|^2$ is equal to

$$(\partial_z e_{wkb}|\partial_z e_{wkb}) - |(\partial_z e_{wkb}|e_{wkb})|^2 + \mathscr{O}\left(h^{-1}\mathrm{e}^{-\frac{1}{Ch}}\right).$$

Let $\Pi_{e_{wkb}}$ and $\Pi_{f_{wkb}}$ be as in Lemma 2.4.3 and note that

$$\begin{aligned}&\|(1-\Pi_{e_{wkb}})\partial_z e_{wkb}\|^2 = \|\partial_z e_{wkb}\|^2 - |(\partial_z e_{wkb}|e_{wkb})|^2 \quad \text{and}\\ &\|(1-\Pi_{f_{wkb}})\partial_z f_{wkb}\|^2 = \|\partial_z f_{wkb}\|^2 - |(\partial_z f_{wkb}|f_{wkb})|^2.\end{aligned} \tag{2.4.13}$$

Hence

$$\begin{aligned}(\partial_z X|\partial_z X) - \frac{|(\partial_z X|X)|^2}{\|X\|^2} =& \|(1-\Pi_{e_{wkb}})\partial_z e_{wkb}\|^2 + \|(1-\Pi_{f_{wkb}})\partial_{\bar z} f_{wkb}\|^2\\ &+\mathscr{O}\left(h^{-1}\mathrm{e}^{-\frac{1}{Ch}} + h^\infty\right).\end{aligned} \tag{2.4.14}$$

Since $\{p,\overline{p}\}(\rho_\pm) = -2i\operatorname{Im} g'(x_\pm)$, it follows by Lemma 2.4.3 and (2.4.14) that

$$(\partial_z X|\partial_z X) - \frac{|(\partial_z X|X)|^2}{\|X\|^2} = \frac{1}{h}\left(\frac{i}{\{p,\overline{p}\}(\rho_+(z))} - \frac{i}{\{p,\overline{p}\}(\rho_-(z))}\right) + \mathscr{O}(1) \tag{2.4.15}$$

Now let us consider the case where $z\in\Omega\cap\Omega^a_\eta$. Similar to Lemma 2.4.3 we get that

$$\begin{aligned}\|(1-\Pi_{e^\eta_{wkb}})\partial_{\tilde z} e^\eta_{wkb}(\cdot,\tilde z)\|^2_{L^2(S^1/\sqrt{\eta},\sqrt{\eta}d\tilde x)} &= \frac{-1}{2\tilde h\operatorname{Im} g'(x_+(z))} + \mathscr{O}(1),\\ \|(1-\Pi_{f^\eta_{wkb}})\partial_{\bar{\tilde z}} f^\eta_{wkb}(\cdot,\tilde z)\|^2_{L^2(S^1/\sqrt{\eta},\sqrt{\eta}d\tilde x)} &= \frac{1}{2\tilde h\operatorname{Im} g'(x_-(z))} + \mathscr{O}(1),\end{aligned}$$

where $|\operatorname{Im} g'(x_\pm(z))| \asymp \sqrt{\eta}$. A rescaling argument, similar to the one in the proof of Proposition 2.1.11, and Corollary 2.1.13 then imply

$$(\partial_z e_0|\partial_z e_0) - |(\partial_z e_0|e_0)|^2 = \frac{-1}{2\tilde h\operatorname{Im} g'(x_+(z))} + \mathscr{O}(\eta^{-2})$$

and similar for $(\partial_{\bar z} f_0|\partial_{\bar z} f_0) - |(f_0|\partial_{\bar z} f_0)|^2$. Hence,

$$(\partial_z X|\partial_z X) - \frac{|(\partial_z X|X)|^2}{\|X\|^2} = \frac{1}{h}\left(\frac{i}{\{p,\overline{p}\}(\rho_+(z))} - \frac{i}{\{p,\overline{p}\}(\rho_-(z))}\right) + \mathscr{O}(\eta^{-2})$$

with $|\{p,\overline{p}\}(\rho_\pm(z))| \asymp \sqrt{\eta}$. The statement on the derivatives of the error estimates follow by the Stationary phase method and the usual rescaling argument. □

2.4.2 – Link with the tunneling effects

We will prove the following result in the light of Proposition 2.2.2.

Proposition 2.4.5. *Let $z \in \Omega \Subset \Sigma$, let $X(z)$ be as in Definition 2.3.2 and let $E_{-+}(z)$ be as in Proposition 2.2.1. Let S be as in Definition 1.2.2. Then,*

$$\left|\partial_z E_{-+}(z) - E_{-+}(z)\frac{(\partial_z X(z)|X(z))}{\|X(z)\|^2} - (e_0|f_0)\right| \leq \mathcal{O}\left(h^\infty \mathrm{e}^{-\frac{S}{h}}\right).$$

Proof of Proposition 2.4.5. Apply the ∂_z derivative to the first equation in (1.2.8),

$$(P_h - z)\partial_z e_0 - e_0 = \partial_z \alpha_0 \cdot f_0 + \alpha_0 \partial_z f_0.$$

Taking the scalar product with f_0 (which is L^2-normalized) then yields

$$(\partial_z e_0|(P_h - z)^* f_0) - (e_0|f_0) = \partial_z \alpha_0 + \alpha_0(\partial_z f_0|f_0).$$

Recall from Proposition 2.2.1 that $E_{-+}(z) = -\alpha_0(z)$ and use the second equation in (1.2.8) to see

$$\partial_z E_{-+}(z) - E_{-+}(z)((\partial_z e_0|e_0) - (\partial_z f_0|f_0)) - (e_0|f_0) = 0. \tag{2.4.16}$$

By Definition 2.3.2 we have the following identity

$$(\partial_z X|X) = \sum_{|j|,|k|<\frac{C_1}{h}} \left(\partial_z \widehat{e}_0(z;j)\overline{\widehat{f}_0(z;k)} + \widehat{e}_0(z;j)\overline{\partial_{\bar{z}}\widehat{f}_0(z;k)}\right)\left(\overline{\widehat{e}_0(z;j)}\widehat{f}_0(z;k)\right).$$

Proposition 2.3.3, Corollary 2.3.4 and the Parseval identity then imply

$$\frac{(\partial_z X|X)}{\|X\|^2} = (\partial_z e_0|e_0) + (f_0|\partial_{\bar{z}} f_0) + \mathcal{O}\left(h^\infty\right). \tag{2.4.17}$$

Note that in the above we also used that e_0 and f_0 are normalized. Since $(f_0|\partial_{\bar{z}} f_0) = -(\partial_z f_0|f_0)$ we conclude by the triangular inequality

$$\left|\partial_z E_{-+}(z) - E_{-+}(z)\frac{(\partial_z X(z)|X(z))}{\|X(z)\|^2} - (e_0|f_0)\right| \leq \mathcal{O}(h^\infty)|E_{-+}(z)|.$$

The statement of the proposition then follows by the estimate $|E_{-+}(z)| = \mathcal{O}\left(\eta^{\frac{1}{4}} h^{\frac{1}{2}} \mathrm{e}^{-\frac{S}{h}}\right)$ given in Proposition 2.2.6. ☐

2.5 | Preparations for the distribution of eigenvalues of P_h^δ

To calculate the intensity measure of Ξ we make use of the following observations:

2.5.1 – Counting zeros

Lemma 2.5.1. *Let $\Omega \subset \mathbb{C}$ be open and convex and let $g, F : \Omega \longrightarrow \mathbb{C}$ be C^∞ such that $g \not\equiv 0$ and*

$$\partial_{\bar{z}} g(z) + \partial_{\bar{z}} F(z) \cdot g(z) = 0 \tag{2.5.1}$$

holds for all $z \in \Omega$. The zeros of g form a discrete set of locally finite multiplicity.The notion of multiplicity here is the same as for holomorphic functions, more details can be found in the proof. Furthermore, for all $\varphi \in \mathscr{C}_0(\Omega)$

$$\left\langle \chi\left(\frac{g}{\varepsilon}\right)\frac{1}{\varepsilon^2}|\partial_z g|^2, \varphi\right\rangle \longrightarrow \sum_{z\in g^{-1}(0)} \varphi(z), \quad \varepsilon \to 0,$$

where $\chi \in \mathscr{C}_0^\infty(\mathbb{C})$ such that $\chi \geq 0$ and $\int \chi(w)L(dw) = 1$ and the zeros are counted according to their multiplicities.

Proof. (2.5.1) implies that

$$e^{F(z)}g(z) \tag{2.5.2}$$

is holomorphic in Ω. g has the same zeros as the holomorphic function (2.5.2). Thus, the zeros of g in Ω form a discrete set and the notion of the multiplicity of the zeros of g is well-defined since we can view the zeros as those of a holomorphic function.

Let $z_0 \in g^{-1}(0)$ have multiplicity n. There exists a neighborhood $W \subset \Omega$ of z_0 such that $W \cap g^{-1}(0) = z_0$. Since $e^{F(z)}g(z)$ is holomorphic, there exists a neighborhood $U \subset \Omega$ of z_0 and a holomorphic function $f : U \to \mathbb{C}$ such that for all $z \in U$

$$f(z) \neq 0, \quad \text{and} \quad e^{F(z)}g(z) = f(z)(z-z_0)^n. \tag{2.5.3}$$

Choose a $\lambda > 0$ such that $|e^{-F(z)}f(z) - e^{-F(z_0)}f(z_0)| < |e^{-F(z_0)}f(z_0)|$ for $|z-z_0| < \lambda$. In this disk we can define a single-valued branch of $\sqrt[n]{e^{-F(z)}f(z)}$.

We take a test function $\varphi \in \mathscr{C}_0(\Omega)$ with

$$\operatorname{supp}\varphi \subset (U \cap W \cap \{z : \, |z-z_0| < \lambda\}) =: N \tag{2.5.4}$$

and consider for $\varepsilon > 0$

$$\left\langle \chi\left(\frac{g}{\varepsilon}\right)\frac{1}{\varepsilon^2}\left|\partial_z g\right|^2, \varphi \right\rangle = \frac{1}{\varepsilon^2}\int\limits_N \chi\left(\frac{g(z)}{\varepsilon}\right)|\partial_z g(z)|^2 \varphi(z) L(dz).$$

Let us perform a change of variables. Define

$$w := g(z) = (z-z_0)^n e^{-F(z)} f(z), \tag{2.5.5}$$

On computes that

$$\begin{aligned} &\partial_z w(z) = (z-z_0)^{n-1} e^{-F(z)}\left(nf(z) + (z-z_0)(\partial_z f(z) - \partial_z F(z) f(z))\right), \\ &\partial_{\overline{z}} w(z_0) = 0. \end{aligned} \tag{2.5.6}$$

Let $r_0 > 0$ be such that $\overline{D(z_0, r_0)} \subset U$, and define

$$C(r_0) := \min_{\underline{z \in D(z_0,r_0)}} |f(z)| \quad \text{and} \quad M(r_0) := \max_{\underline{z \in D(z_0,r_0)}} |\partial_z f(z) - \partial_z F(z) f(z)|.$$

By (2.5.3) it follows that $C(r_0) > 0$ and we may assume that $M(r_0) > 0$ since else, it follows immediately from (2.5.6) that $\partial_z w(z) \neq 0$ for all $z \in D(z_0, r_0)\backslash\{z_0\}$.

Let $0 < r < \min\{C(r_0)n/(2M(r_0)), r_0\}$, the triangular inequality applied to (2.5.6) then implies that $\partial_z w(z) \neq 0$ for all $z \in D(z_0, r)\backslash\{z_0\}$. The implicit function theorem implies that we can invert equation (2.5.5) for z in the disk $D(z_0, r)\backslash\{z_0\}$, and w in the n-fold covering surface of $w(D(z_0, r)\backslash\{z_0\})$. Thus, if we denote the domain on each leaf of the covering by B_k, for $k = 1, \dots, n$, as a subset of $\mathbb{C}$, and the respective branch of g by g_k we get for $\varepsilon > 0$ small enough

$$\left\langle \chi\left(\frac{g}{\varepsilon}\right)\frac{1}{\varepsilon^2}\left|\partial_z g\right|^2, \varphi \right\rangle = \sum_{k=1}^n \frac{1}{\varepsilon^2}\int_{B_k} \varphi(g_k^{-1}(w))\chi\left(\frac{w}{\varepsilon}\right)(1 + \mathscr{O}(w^2))L(dw),$$

with $g_k^{-1}(0) = z_0$. In the above we used that

$$L(dw) = \left(|\partial_z g(z)|^2 - |\partial_{\overline{z}} g(z)|^2\right) Ld(z)$$

and the $\overline{\partial}$-equation (2.5.1) which implies

$$|\partial_{\overline{z}} g(z)|^2 = |\partial_{\overline{z}} F(z) g(z)|^2 = \mathscr{O}(w^2).$$

Thus we can conclude

$$\left\langle \chi\left(\frac{g}{\varepsilon}\right)\frac{1}{\varepsilon^2}\left|\partial_z g\right|^2, \varphi \right\rangle \longrightarrow \sum_{k=1}^n \varphi(z(0)) = n\varphi(z_0), \quad \text{for } \varepsilon \to 0. \tag{2.5.7}$$

Since g has at most countably many zeros in Ω, there exists some index set $I \subset \mathbb{N}$ such that we can denote the set of zeros of g in Ω by $\{z_i\}_{i\in I} := g^{-1}(0) \cap \Omega$. Furthermore, let $m(i)$ for all $i \in I$ denote the multiplicity of the respective zero z_i.
For each zero z_i we can construct a neighborhood N_i, as above, such that for a test function with support in N_i we have the convergence as in (2.5.7). By potentially shrinking the N_i we can gain $N_i \cap N_j = \emptyset$ for $i \neq j$. Consider the following locally finite open covering of Ω

$$\Omega = \left(\bigcup_{i\in I} N_i\right) \cup (\Omega \backslash \{z_i : \, i \in I\}).$$

Let $\{\chi_i\}_{i\in I\cup\{0\}}$ be a partition of unity subordinate to this open covering such that

$$1 = \sum_{i\in I} \chi_i + \chi_0.$$

Here $\chi_i \in \mathscr{C}_0^\infty(N_i)$ and $\chi_i \equiv 1$ in a neighborhood of z_i for all $i \in I$. Furthermore, $\chi_0 \in \mathscr{C}^\infty(\Omega)$ and $z_i \notin \operatorname{supp}\chi_0$ for all $i \in I$. Let $\varphi \in \mathscr{C}_0(\Omega)$ be an arbitrary test function. By (2.5.7) we have for $\varepsilon \to 0$

$$\left\langle \chi\left(\frac{g}{\varepsilon}\right)\frac{1}{\varepsilon^2}|\partial_z g|^2, \varphi\right\rangle = \sum_{i\in I}\left\langle \chi\left(\frac{g}{\varepsilon}\right)\frac{1}{\varepsilon^2}|\partial_z g|^2, \chi_i\varphi\right\rangle \to \sum_{i\in I} m(i)\chi_i(z_i)\varphi(z_i).$$

Since $g(z) \neq 0$ for all $z \in \operatorname{supp}\chi_0$ we have for $\varepsilon > 0$ small enough

$$\left\langle \chi\left(\frac{g}{\varepsilon}\right)\frac{1}{\varepsilon^2}|\partial_z g|^2, \chi_0\varphi\right\rangle = 0$$

and we can conclude the statement of the Lemma. □

2.5.2 – An implicit function theorem

Lemma 2.5.2. *Let $R > 0$ and $a > c \geq 0$ be constants. Let $D(0,R) \subset \mathbb{C}$ be the open disk of radius R centered at 0 and let $g, f : D(0,R) \longrightarrow \mathbb{C}$ be holomorphic such that*

$$\|g\|_\infty \leq c, \quad \textit{and for all } z \in D(0,R) : \, \partial_z f(z) = a + g(z). \tag{2.5.8}$$

Assume that

$$\xi \in D\big(f(0), (a-c)R\big) \subset \mathbb{C}.$$

Then the equation

$$f(z) = \xi$$

has exactly one solution $z = z(\xi) \in D(0,R)$ and it depends holomorphically on ξ.

Proof. For $z \in D(0,R)$

$$f(z) = \int_0^z \big(a + g(w)\big)\,dw + f(0) = az + f(0) + G(z),$$

where $G(z) := \int_0^z g(w)dw$. Now let us consider the equation

$$az + f(0) - \xi = 0.$$

The unique solution lies in the disk $D(0,R)$ since

$$\frac{|\xi - f(0)|}{a} < \frac{|a-c|}{a}R < R.$$

Now consider for $\varepsilon > 0$ and for $z \in D(0, R-\varepsilon)$ the equation

$$f(z) - \xi = az + f(0) - \xi + G(z) = 0.$$

Recall that $\xi \in D\big(f(0),(a-c)R\big)$ which implies that there exists a $\varepsilon(\xi)>0$ such that $|\xi - f(0)| \leq (a-c)(R-\varepsilon(\xi))$. Thus for all $\varepsilon < \varepsilon(\xi)$

$$|az + f(0)-\xi| \geq |az| - |f(0)-\xi| > a|z| - (a-c)(R-\varepsilon)$$

and, using that $|G(z)| \leq c|z|$, we may conclude that for $|z| = R-\varepsilon$

$$|G(z)| < |az+f(0)-\xi|.$$

By Rouché's theorem we have that $az+f(0)-\xi$ and $f(z)-\xi$ have the same number of zeros in the disk $D(0,R-\varepsilon)$. We also see that $f(z)-\xi$ has no zero in $D(0,R)\backslash D(0,R-\varepsilon)$ and the result follows. □

Proposition 2.5.3. *Let $a>c\geq 0$ be constants, $n\in\mathbb{N}$, let $\Omega\subset\mathbb{C}^n$ be open, bounded and of the form*

$$\Omega = \{z=(z',z_n)\in\mathbb{C}^n \,:\, z'\in\Omega',\ |z_n|<R_{z'}\}$$

where $R_{z'}>0$ is continuous in z'. Furthermore, assume that

- *$g,F:\Omega\longrightarrow\mathbb{C}$ are holomorphic such that*

$$\|g\|_\infty \leq c, \quad \text{and for all } z\in\Omega:\ \partial_{z_n}F(z) = a+g(z), \tag{2.5.9}$$

- *$\Gamma \Subset \Omega'$ is open so that $\inf\limits_{z'\in\Gamma} R_{z'} \geq \text{const.}>0$,*
- *$\xi \in \bigcap_{z'\in\Gamma} D\big(F(z',0),(a-c)R_{z'}\big)\subset\mathbb{C}$.*

Then, when $z'\in\Gamma$, the equation

$$F(z',z_n)=\xi$$

has exactly one solution $z_n(z',\xi)\in D(0,R_{z'})$ and it depends holomorphically on ξ and on $z'\in\Gamma$.

Proof. Lemma 2.5.2 implies the existence an uniqueness of the solutions $z_n(z',\xi)$ in each disk $D(0,R_{z'})$. By (2.5.9) it follows that

$$\frac{\partial F}{\partial z_n}(z',z_n(z',\xi)) \neq 0$$

for all $z'\in\Gamma$ and all $\xi\in D\big(F(z',0),(a-c)(R_{z'}-\lambda)\big)$. Hence, the implicit function theorem implies that $z_n(z',\xi)$ depends holomorphically on ξ and z'. □

2.6 | A formula for the intensity measure of the point process of eigenvalues of P_h^δ

We prove the following formula for the intensity measure of Ξ (cf (1.2.10)):

Proposition 2.6.1. *Let $h^{2/3}\ll\eta<\text{const.}$ and let $\Omega := \Omega_\eta^a \Subset \Sigma$. Let $C>0$ and let $C_1>0$ be as in (1.1.10) such that $C-C_1>0$ is large enough. Let δ be as in Hypothesis 1.2.6 with $\kappa>4$, define $N:=(2\lfloor C_1/h\rfloor+1)^2$ and let $B(0,R)\subset\mathbb{C}^N$ be the ball of radius $R:=Ch^{-1}$ centered at zero. For $z\in\Omega$ let $X(z)$ be as in Definition 2.3.2, let $E_{-+}(z)$ be as in Proposition 2.2.1 and let e_0 and f_0 be as in (1.2.6) and (1.2.9). There exist functions*

$$\begin{aligned}\Psi(z;h,\delta) &= (\partial_z X|\partial_z X) - \frac{1}{\|X\|^2}|(\partial_z X|X)|^2 \\ &\quad + \delta^{-2}\left|(e_0|f_0)(1+\mathcal{O}(h^\infty)) + \mathcal{O}\big(\eta^{1/4}\delta^2h^{-7/2}\big)\right|^2 + \mathcal{O}\big(\delta^3h^{-3}\big),\end{aligned} \tag{2.6.1}$$

$$\Theta(z;h,\delta) = \frac{|E_{-+}(z)+\mathcal{O}\big(\delta^2\eta^{-1/4}h^{-5/2}\big)|^2}{\delta^2\|X(z)\|^2}, \tag{2.6.2}$$

and $D>0$ and $\tilde{C}>0$ such that for all $\varphi\in\mathscr{C}_0(\Omega)$ and for $h>0$ small enough

$$\mathbb{E}\left[\,\Xi(\varphi)\mathbb{1}_{B(0,R)}\,\right]=\int\varphi(z)\frac{1+\mathscr{O}\left(\delta\eta^{-1/4}h^{-3/2}\right)}{\pi\|X\|^2}\Psi(z;h,\delta)\mathrm{e}^{-\Theta(z;h,\delta)}L(dz)$$
$$+\mathscr{O}\left(\mathrm{e}^{-\frac{D}{h^2}}\right).$$

Here, $\mathscr{O}\left(\eta^{-1/4}\delta h^{-3/2}\right)$ is independent of φ and $\mathscr{O}\left(\mathrm{e}^{-\frac{D}{h^2}}\right)$ means $\langle T_h,\varphi\rangle$ where $T_h\in\mathscr{D}'(\mathbb{C})$ such that $|\langle T_h,\varphi\rangle|\leq C\|\varphi\|_\infty\mathrm{e}^{-\frac{D}{h^2}}$ for all $\varphi\in\mathscr{C}_0(\Omega)$ where C and D is independent of h, δ, η and φ. Moreover, the estimates in (2.6.1) and (2.6.2) are stable under application of $\eta^{-\frac{n+m}{2}}h^{n+m}\partial_z^n\partial_{\overline{z}}^m$.

Proof. **Step I** Recall from Sections 2.2 and 2.3 that $\sigma(P_h^\delta)=(E_{-+}^\delta)^{-1}(0)$, thus Ξ (cf. Definition 1.2.10) satisfies

$$\Xi=\sum_{z\in(E_{-+}^\delta)^{-1}(0)}\delta_z.$$

It has been shown in [67], that $E_{-+}^\delta(z)$ satisfies a $\overline{\partial}$-equation, i.e. there exists a smooth function $f^\delta:\Omega\to\mathbb{C}$ such that

$$\partial_{\overline{z}}E_{-+}^\delta(z)+f^\delta(z)E_{-+}^\delta(z)=0.$$

This implies that the zeros of $E_{-+}^\delta(z)$ are isolated and countable and we may use the same notion of multiplicity as for holomorphic functions. In particular, $E_{-+}^\delta(z)$ satisfies condition (2.5.1). Let χ be as in Lemma 2.5.1, then by Lemma 2.5.1, Fubini's theorem and the dominated convergence theorem we have

$$\mathbb{E}\left[\sum_{z\in(E_{-+}^\delta)^{-1}(0)}\varphi(z)\,\mathbb{1}_{B(0,R)}\right]=\lim_{\varepsilon\to 0}\int\varphi(z)\left(\int_{B(0,R)}D(z,\alpha)L(d\alpha)\right)L(dz),$$
$$\text{where } D(z,\alpha)=\pi^{-N}\chi\left(\frac{E_{-+}^\delta(z,\alpha)}{\varepsilon}\right)\frac{1}{\varepsilon^2}\left|\partial_zE_{-+}^\delta(z,\alpha)\right|^2\mathrm{e}^{-\alpha\overline{\alpha}}. \tag{2.6.3}$$

Step II Next we give an estimate on $\partial_zE_{-+}^\delta(z)$. By (2.3.2)

$$\partial_zE_{-+}^\delta(z)=\partial_zE_{-+}(z)-\delta\left(\partial_zX(z)\cdot\alpha+\partial_zT(z,\alpha)\right), \tag{2.6.4}$$

where the derivative ∂_z acts on $X(z)$ component wise and the dot-product $\partial_zX(z)\cdot\alpha$ is bilinear. To estimate $\partial_zT(z,\alpha)$, recall (2.3.3) and consider the derivative

$$\partial_zE_-Q_\omega(EQ_\omega)^nE_+=(\partial_zE_-)Q_\omega(EQ_\omega)^nE_+$$
$$+E_-Q_\omega\left[\sum_{j=1}^n(EQ_\omega)^{j-1}(\partial_zE)Q_\omega(EQ_\omega)^{n-j}\right]E_++E_-Q_\omega(EQ_\omega)^n(\partial_zE_+),$$

with the convention $(EQ_\omega)^0=1$. Recall the Grushin problem from Proposition 2.2.1 and take the derivative with respect to z of the relation $\mathscr{E}(z)\mathscr{P}(z)=1$ to obtain

$$\partial_z\mathscr{E}(z)+\mathscr{E}(z)(\partial_z\mathscr{P}(z))\mathscr{E}(z)=0.$$

A direct calculation yields

$$\partial_zE=-E(\partial_z(P_h-z))E-E_+(\partial_zR_+)E-E(\partial_zR_-)E_-$$
$$=E^2-E_+(\partial_zR_+)E-E(\partial_zR_-)E_-.$$

Recall the definition of R_+ and R_- given in (2.2.1). By the estimates on the z- and $\overline{z}$- derivatives of e_0 and f_0 given in Lemma 2.1.11, we conclude that

$$\|\partial_zR_+\|_{H^1\to\mathbb{C}},\|\partial_zR_-\|_{\mathbb{C}\to L^2}=\mathscr{O}\left(\eta^{1/2}h^{-1}\right).$$

Similarly, we have the same estimates on $\|\partial_z E_+\|_{\mathbb{C}\to L^2}$ and $\|\partial_z E_-\|_{H^1\to\mathbb{C}}$. Thus, since $\|E(z)\|_{L^2\to H^1} = \mathscr{O}((h\sqrt{\eta})^{-1/2})$ and $\|E_\pm\| = \mathscr{O}(1)$, we have

$$\|\partial_z E\|_{L^2\to H^1} = \mathscr{O}(\eta^{1/4}h^{-3/2}).$$

Putting all of this together, we get that the series of $\partial_z T(z,\alpha)$ converges again geometrically and we gain the estimate

$$\partial_z T(z,\alpha) = \mathscr{O}\big(\eta^{1/4}\delta h^{-7/2}\big). \tag{2.6.5}$$

Analogously, we conclude for all $\beta\in\mathbb{N}^2$

$$\eta^{-\frac{|\beta|}{2}} h^{|\beta|}\partial^\beta_{z\overline{z}} T(z,\alpha) = \mathscr{O}\big(\eta^{-1/4}\delta h^{-5/2}\big). \tag{2.6.6}$$

Thus,

$$\partial_z E^\delta_{-+}(z) = \partial_z E_{-+}(z) - \delta\partial_z X(z)\cdot\alpha + \mathscr{O}\big(\eta^{1/4}\delta h^{-7/2}\big).$$

Step III Consider the integral (2.6.3) and choose vectors $e_1, e_2, \dots \in \mathbb{C}^N$ as a basis of the α-space such that $e_1 = \overline{X}/\|\overline{X}\|$ and such that e_1, e_2 and $\overline{X}/\|\overline{X}\|, \partial_z\overline{X}$ span the same space: Therefore, we perform a unitary transformation in the α-space such that with a slight abuse of notion

$$\alpha = \alpha_1\frac{\overline{X(z)}}{\|X(z)\|} + \alpha_2 b\left(\frac{\overline{\partial_z X(z)}}{\|\partial_z X(z)\|} - \frac{\overline{(\partial_z X(z)|X(z))X(z)}}{\|\partial_z X(z)\|\,\|X(z)\|^2}\right) + \alpha^\perp, \tag{2.6.7}$$

where $\alpha_1,\alpha_2\in\mathbb{C}$ and $\alpha^\perp\in\mathbb{C}^{N-3}$ and $b>0$ is a factor of normalization,

$$b = \frac{\|\partial_z X(z)\|\,\|X(z)\|}{\sqrt{\|\partial_z X(z)\|^2\,\|X(z)\|^2 - |(\partial_z X(z)|X(z))|^2}}. \tag{2.6.8}$$

This change of variables is well defined by Lemma 2.4.1. In the following we will also use the notation $(\alpha_1,\alpha_2,\alpha^\perp) = (\alpha_1,\alpha')$. This choice of basis yields by (2.3.3) and (2.3.2)

$$E^\delta_{-+}(z) = E_{-+}(z) - \delta\|X(z)\|\alpha_1 + \mathscr{O}\big(\eta^{-1/4}\delta^2 h^{-5/2}\big) \tag{2.6.9}$$

and by (2.6.4), (2.6.7), (2.6.8)

$$\begin{aligned}\partial_z E^\delta_{-+}(z) = \partial_z E_{-+}(z) &- \delta\frac{(\partial_z X(z)|X(z))}{\|X(z)\|}\alpha_1 \\ &- \delta\left(\|\partial_z X(z)\|^2 - \frac{|(\partial_z X(z)|X(z))|^2}{\|X(z)\|^2}\right)^{\frac12}\alpha_2 + \mathscr{O}\big(\eta^{1/4}\delta^2 h^{-7/2}\big).\end{aligned} \tag{2.6.10}$$

Now let us split the ball $B(0,R)$, $R = Ch^{-1}$, into two pieces: pick $C_0>0$ such that $0<C_1<C_0<C$ and define $R_0 := C_0 h^{-1}$. Then we shall consider one piece such that $\|\alpha'\|_{\mathbb{C}^{N-1}} < R_0$ and the other such that $\|\alpha'\|_{\mathbb{C}^{N-1}} > R_0$. Hence, (2.6.3) is equal to

$$\begin{aligned}&\lim_{\varepsilon\to0}\int\varphi(z)\int_{\substack{B(0,R)\\ \|\alpha'\|_{\mathbb{C}^{N-1}}<R_0}} D(z,\alpha)L(d\alpha)L(dz) + \lim_{\varepsilon\to0}\int\varphi(z)\int_{\substack{B(0,R)\\ \|\alpha'\|_{\mathbb{C}^{N-1}}>R_0}} D(z,\alpha)L(d\alpha)L(dz)\\ &=: I_1(\varphi) + I_2(\varphi).\end{aligned} \tag{2.6.11}$$

Step IV In this step we will calculate $I_1(\varphi)$ of (2.6.11). There we perform a change of variables such that $\beta := E^\delta_{-+}(z,\alpha)$ is one of them. Due to (2.6.9) it is natural to express α_1 as a function of β and α'. To this purpose we will apply Proposition 2.5.3 to the function $E^\delta_{-+}(z,\alpha)$:

$E^{\delta}_{-+}(z,\alpha_1,\alpha')$ is holomorphic in α in ball of radius $R = Ch^{-1}$ centered at 0. Here, α plays the role of z in the Proposition, in particular α_1 plays the role of z_n. Recall (2.3.2) and note that since $T(z,\alpha) = \mathcal{O}\big(\eta^{-1/4}\delta h^{-5/2}\big)$ (cf. (2.3.3)) we can conclude by the Cauchy inequalities that

$$\partial_{\alpha_1}\delta T(z,\alpha) = \mathcal{O}\big(\eta^{-1/4}\delta^2 h^{-3/2}\big)$$

which implies

$$\partial_{\alpha_1} E^{\delta}_{-+}(z,\alpha_1,\alpha') = -\delta\|X(z)\| + \mathcal{O}\big(\eta^{-1/4}\delta^2 h^{-3/2}\big). \tag{2.6.12}$$

By Proposition 2.3.3 we have that $\|X(z)\| = 1+\mathcal{O}(h^{\infty})$ which implies that

$$\partial_{\alpha_1} E^{\delta}_{-+}(z,\alpha_1,\alpha') = -\delta\left(1+\mathcal{O}\big(h^{\infty}+\eta^{-1/4}\delta h^{-3/2}\big)\right).$$

Hence, $E^{\delta}_{-+}(z,\alpha)$ satisfies the assumptions of Proposition 2.5.3. Since we restricted α' to $\|\alpha'\|_{\mathbb{C}^{N-1}} < R_0$ and since

$$|\alpha_1| < R^2 - \|\alpha'\|_{\mathbb{C}^{N-1}} =: R_{\alpha'},$$

it follows by Proposition 2.5.3 that for

$$\beta \in \bigcap_{\|\alpha'\|_{\mathbb{C}^{N-1}}<R_0} D\Big(E^{\delta}_{-+}(z,0,\alpha'), r_{\alpha'}\Big) \tag{2.6.13}$$

with

$$r_{\alpha'} \geq \delta\left(1+\mathcal{O}\big(h^{\infty}+\eta^{-1/4}\delta h^{-3/2}\big)\right)\frac{\sqrt{C^2-C_0^2}}{h} \geq \frac{\delta h^{-1}}{\asymp 1} > 0. \tag{2.6.14}$$

and $h>0$ small enough, $\beta = E^{\delta}_{-+}(z,\alpha_1,\alpha')$ has exactly one solution $\alpha_1(\beta,\alpha')$ in the disk $D(0,R_{\alpha'})$ and it depends holomorphically on β and α'. More precisely,

$$\alpha_1(\beta,\alpha') = \frac{-\beta + E_{-+}(z) + \mathcal{O}\big(\eta^{-1/4}\delta^2 h^{-5/2}\big)}{\delta\|X(z)\|}. \tag{2.6.15}$$

Furthermore,

$$L(d\alpha) = |\partial_{\alpha_1}E^{\delta}_{-+}|^{-2}L(d\beta)L(d\alpha').$$

Since the support of χ is compact, we can restrict our attention to β and $E^{\delta}_{-+}(z,0,\alpha')$ in a small disk of radius $\varepsilon>0$ centered at 0. By choosing $\varepsilon < \delta h^{-1}/C$, $C>0$ large enough, as in (2.6.14) we see that $\beta, E^{\delta}_{-+}(z,0,\alpha') \in D(0,\varepsilon)$ implies (2.6.13). By performing this change of variables and by picking $\varepsilon>0$ small enough as above, we get

$$I_1(\varphi) = \lim_{\varepsilon\to 0}\int \varphi(z)\left\{\int_{\mathbb{C}} \chi\left(\frac{\beta}{\varepsilon}\right)\frac{1}{\varepsilon^2}\Lambda(\beta,z)L(d\beta)\right\}L(dz), \tag{2.6.16}$$

where $\Lambda(\beta,z)$ depends smoothly on z and on β and, using (2.6.10), is given by

$$\begin{aligned}\Lambda(\beta,z) :=& \pi^{-N}\int_{\|\alpha'\|_{\mathbb{C}^{N-1}}<R_0} \mathbb{1}_{B(0,R)}(\alpha_1,\alpha')\left|\partial_{\alpha_1}E^{\delta}_{-+}(\alpha_1,\alpha',z)\right|^{-2} \\ &\cdot\left|A(\alpha,z) - \beta\frac{(\partial_z X(z)|X(z))}{\|X(z)\|^2} - B(z)\alpha_2 + \mathcal{O}\big(\eta^{1/4}\delta^2 h^{-7/2}\big)\right|^2 \\ &\cdot\exp\left\{-\alpha'\overline{\alpha'} - \left|\frac{-\beta+E_{-+}(z)+\mathcal{O}\big(\eta^{-1/4}\delta^2h^{-5/2}\big)}{\delta\|X(z)\|}\right|^2\right\}L(d\alpha'),\end{aligned} \tag{2.6.17}$$

where where $\alpha_1 = \alpha_1(\beta,\alpha',z)$ and $A(\alpha,z), B(z)$ are defined as follows:

$$\begin{aligned}A(\alpha,z) :=& \partial_z E_{-+}(z) - \frac{(\partial_z X(z)|X(z))}{\|X(z)\|^2}\left(E_{-+}(z)+\mathcal{O}\big(\eta^{-1/4}\delta^2h^{-5/2}\big)\right) \\ &+\mathcal{O}\big(\eta^{1/4}\delta^2h^{-7/2}\big) \\ =& (e_0|f_0)(1+\mathcal{O}(h^{\infty})) + \mathcal{O}\big(\eta^{1/4}\delta^2h^{-7/2}\big) \\ =& \mathcal{O}\left(\eta^{3/4}h^{-\frac{1}{2}}\mathrm{e}^{-\frac{\asymp\eta^{3/2}}{h}}\right) + \mathcal{O}\big(\eta^{1/4}\delta^2h^{-7/2}\big).\end{aligned} \tag{2.6.18}$$

The second identity for A is due to Proposition 2.4.5 and the following estimate

$$\left|\frac{(\partial_z X(z)|X(z))}{\|X(z)\|^2}\right| \leq \frac{\|\partial_z X(z)\|}{\|X(z)\|} = \left(1+\mathcal{O}\left(h^\infty\right)\right)\mathcal{O}\left(\eta^{1/2}h^{-1}\right) = \mathcal{O}\left(\eta^{1/2}h^{-1}\right)$$

which follows from Propositions 2.3.3 and 2.3.4. In the last line we used Proposition 2.2.6 together with (2.1.5). Furthermore, recall by Step II and Step III that $A(\alpha,z)$ is holomorphic in α.

Similarly, we define

$$B(z) := \delta\left(\|\partial_z X(z)\|^2 - \frac{|(\partial_z X(z)|X(z))|^2}{\|X(z)\|^2}\right)^{\frac{1}{2}} = \mathcal{O}\left(\eta^{-1/4}\delta h^{-\frac{1}{2}}\right). \tag{2.6.19}$$

The estimate in (2.6.19) follows from Proposition 2.4.1.

Remark 2.6.2. It follows from Proposition 2.4.5, Proposition 2.2.6, Proposition 2.3.4 and from (2.6.6) that

$$\begin{aligned}
&\eta^{-\frac{n+m}{2}}h^{n+m}\partial_z^n\partial_{\bar z}^m A(z) = \mathcal{O}\left(\eta^{3/4}h^{-\frac{1}{2}}\mathrm{e}^{-\frac{\asymp\eta^{3/2}}{h}}\right) + \mathcal{O}\left(\eta^{1/4}\delta^2 h^{-7/2}\right),\\
&\eta^{-\frac{n+m}{2}}h^{n+m}\partial_z^n\partial_{\bar z}^m B(z) = \mathcal{O}\left(\eta^{-1/4}\delta h^{-\frac{1}{2}}\right).
\end{aligned} \tag{2.6.20}$$

Since $\Lambda(\beta,z)$ is continuous in β, the dominated convergence theorem shows that

$$I_1(\varphi) = \int \varphi(z)\Lambda(0,z)L(dz).$$

Next, let us look at the indicator function $\mathbb{1}_{B(0,R)}(\alpha_1(\beta,\alpha',z),\alpha')$ for $\|\alpha'\| < R_0$: By (2.6.15) we have

$$|\alpha_1(0,\alpha')| = \frac{\left|E_{-+}(z) + \mathcal{O}\left(\delta^2 h^{-5/2}\right)\right|}{\delta\|X(z)\|}.$$

Thus, $\mathbb{1}_{B(0,R)}(\alpha_1(0,\alpha',z),\alpha') = 1$ if $|\alpha_1(0,\alpha')|^2 \leq R^2 - R_0^2 = \frac{\tilde C^2}{h^2}$, $\|\alpha'\| < R_0^2$ and if $R^2 - R_0^2 < |\alpha_1(0,\alpha')|^2 < R^2$, $\|\alpha'\| < R_0^2 - |\alpha_1(0,\alpha')|^2$, and $\mathbb{1}_{B(0,R)}(\alpha_1(0,\alpha',z),\alpha') = 0$ if $R^2 \leq |\alpha_1(0,\alpha')|^2$, with $\tilde C^2 := C^2 - C_0^2$. Hence, we split $\Lambda(0,z)$ into

$$\begin{aligned}
\Lambda(0,z) &= \Lambda(0,z)\left(\mathbb{1}_{\left\{\sqrt{\Theta(z;h,\delta)}\leq\frac{\tilde C}{h}\right\}}(z) + \mathbb{1}_{\left\{\frac{\tilde C}{h}<\sqrt{\Theta(z;h,\delta)}<R\right\}}(z)\right)\\
&=: \Lambda_1(0,z) + \Lambda_2(0,z),
\end{aligned} \tag{2.6.21}$$

where

$$\Theta(z;h,\delta) := \frac{|E_{-+}(z) + \mathcal{O}\left(\delta^2\eta^{-1/4}h^{-5/2}\right)|^2}{\delta^2\|X(z)\|^2}.$$

We start by treating Λ_1. Note that the function

$$\{\|\alpha'\|_{\mathbb{C}^{N-1}} < R_0\} \ni \alpha' \longmapsto \exp\left\{-\left|\alpha_1(0,\alpha',z)\right|^2\right\} \in [0,1]$$

is continuous, bounded and recall that (2.6.15) holds for all $\alpha' \in \{\|\alpha'\|_{\mathbb{C}^{N-1}} < R_0\}$. Furthermore, note that all factors in the integral (2.6.17) are positive. Since the ball $\{\|\alpha'\|_{\mathbb{C}^{N-1}} < R_0\}$ is simply connected the intermediate value theorem yields

$$\begin{aligned}
\Lambda_1(0,z) =&\pi^{-N}\mathbb{1}_{\left\{\sqrt{\Theta(z;h,\delta)}\leq\frac{\tilde C}{h}\right\}}(z)\left|\delta\|X(z)\| + \mathcal{O}\left(\eta^{-1/4}\delta^2 h^{-3/2}\right)\right|^{-2}\\
&\cdot\exp\{-\Theta(z;h,\delta)\}\int\limits_{\|\alpha'\|_{\mathbb{C}^{N-1}}<R_0}|A(\alpha,z) - \delta B(z)\alpha_2|^2\,\mathrm{e}^{-\alpha'\overline{\alpha'}}L(d\alpha').
\end{aligned} \tag{2.6.22}$$

Here we also applied (2.6.12). Before we can further simplify (2.6.22), let us prove the following technical Lemma:

Lemma 2.6.3. *Let $h > 0$, let $C_0, C_1 > 0$ and let $N := (2\lfloor \frac{C_1}{h} \rfloor + 1)^2$. Let $n \in \mathbb{N}^{N-1}, m \in \mathbb{N}^{N-1}$, let $R_0 = C_0/h$ and let $\alpha \in \mathbb{C}^N$. If $C_0 > C_1 > 0$ are large enough and such that*

$$\ln\left(2 + \frac{\mathrm{e}R_0^2}{N-2}\right) < \frac{R_0^2}{2(N-2)},$$

then, for $h > 0$ small enough, there exists a constant $D_{n,m} =: D > 0$ such that

$$\left|\pi^{1-N} \int_{\|\alpha'\|_{\mathbb{C}^{N-1}} \geq R_0} \alpha'^n \overline{\alpha'}^m \mathrm{e}^{-\alpha'\overline{\alpha'}} L(d\alpha')\right| = \mathcal{O}\left(\mathrm{e}^{-\frac{D}{h^2}}\right).$$

Proof. Define

$$2u := \begin{cases} |n| + |m|, \text{ if it is even} \\ |n| + |m| + 1, \text{ else} \end{cases}$$

and notice

$$\begin{aligned} &\left|\pi^{1-N} \int_{\|\alpha'\|_{\mathbb{C}^{N-1}} \geq R_0} \alpha'^n \overline{\alpha'}^m \mathrm{e}^{-\alpha'\overline{\alpha'}} L(d\alpha')\right| \\ &\leq \pi^{1-N} \left|S^{2N-3}\right| \int_{R_0}^{\infty} r^{2u+2N-3} \mathrm{e}^{-r^2} dr = \frac{2}{(N-2)!} \int_{R_0^2}^{\infty} \tau^{u+N-2} \mathrm{e}^{-\tau} d\tau. \end{aligned}$$

Repeated partial integration then yields

$$\frac{2}{(N-2)!} \mathrm{e}^{-R_0^2} \sum_{i=0}^{u+N-2} \binom{u+N-2}{i} (u+N-2-i)! R_0^{2i}. \tag{2.6.23}$$

Using Stirling's formula one gets that (2.6.23)$\leq$

$$\begin{aligned} &\frac{\mathrm{e}\sqrt{(u+N-2)}}{(N-2)!} \mathrm{e}^{-R_0^2} \sum_{i=0}^{u+N-2} \binom{u+N-2}{i} \left(\frac{u+N-2}{\mathrm{e}}\right)^{u+N-2-i} R_0^{2i} \\ &\leq \frac{\mathrm{e}\sqrt{(u+N-2)}}{\sqrt{2\pi(N-2)}} \mathrm{e}^{-R_0^2} \left(\frac{\mathrm{e}}{N-2}\right)^{N-2} \left(R_0^2 + \frac{u+N-2}{\mathrm{e}}\right)^{u+N-2} \\ &= \mathrm{e}^{-R_0^2} \frac{\mathrm{e}}{\sqrt{2\pi}} \sqrt{1 + \frac{u}{N-2}} \left(\frac{R_0^2 \mathrm{e}}{N-2} + 1 + \frac{u}{N-2}\right)^{N-2} \left(R_0^2 + \frac{u+N-2}{\mathrm{e}}\right)^u. \end{aligned}$$

Since $u/(N-2)$ is bounded for $h > 0$ small, it remains to consider

$$\exp\left\{-R_0^2 + (N-2)\ln\left(\frac{R_0^2 \mathrm{e}}{N-2} + 1 + \frac{u}{N-2}\right) + u \ln\left(R_0^2 + \frac{u+N-2}{\mathrm{e}}\right)\right\}. \tag{2.6.24}$$

However, there exists a $1 > \kappa > 0$ such that

$$-R_0^2 + (N-2)\ln\left(\frac{R_0^2 \mathrm{e}}{N-2} + 1 + \frac{u}{N-2}\right) \leq -R_0^2 \kappa = -\frac{C_0^2}{h^2},$$

which implies that (2.6.24) is dominated by

$$\exp\left\{-\frac{C_0^2}{h^2}\left(\kappa - \frac{h^2}{\mathcal{O}(1)} \ln(h)\right)\right\},$$

and we conclude the statement of the Lemma for $h > 0$ small enough. ☐

Let us return to (2.6.22): We are interested in the integral

$$\pi^{-N} \int\limits_{\|\alpha'\|_{\mathbb{C}^{N-1}} < R_0} |A - B\alpha_2|^2 \exp\left\{-\alpha'\overline{\alpha'}\right\} L(d\alpha'). \tag{2.6.25}$$

We will investigate each term of (2.6.25) separately. Since B is constant in α and since

$$\int |\alpha_2|^2 \exp(-\alpha'\overline{\alpha'}) L(d\alpha') = \pi^{N-1},$$

we conclude, by Lemma 2.6.3 for $C_0 > C_1 > 0$ large enough and $h > 0$ small enough, that there exists a constant $D > 0$ such that

$$\pi^{-N} \int\limits_{\|\alpha'\|_{\mathbb{C}^{N-1}} < R_0} |B\alpha_2|^2 \mathrm{e}^{-\alpha'\overline{\alpha'}} L(d\alpha') = \pi^{-1}|B|^2 + \mathcal{O}\left(\eta^{-\frac{1}{2}}\delta^2 h^{-1} \mathrm{e}^{-\frac{D}{h^2}}\right).$$

The mean value theorem, (2.6.18) and Lemma 2.6.3 imply that there exists a constant $D > 0$ (not necessarily the same as above) such that

$$\pi^{-N} \int\limits_{\|\alpha'\|_{\mathbb{C}^{N-1}} < R_0} |A|^2 \exp\left\{-\alpha'\overline{\alpha'}\right\} L(d\alpha') = \pi^{-1}|A|^2 + \mathcal{O}\left(\mathrm{e}^{-\frac{D}{h^2}}\right).$$

Note that after the equality sign we have $A = A(\tilde{\alpha}', z)$ for an $\tilde{\alpha}' \in B(0, R_0)$ given by the mean value theorem. Next, since (2.6.19) is independent of α,

$$\pi^{-N} \int\limits_{\|\alpha'\|_{\mathbb{C}^{N-1}} < R_0} \overline{A} B\alpha_2 \mathrm{e}^{-\alpha'\overline{\alpha'}} L(d\alpha') = \pi^{-N} B \int\limits_{\|\alpha'\|_{\mathbb{C}^{N-1}} < R_0} \overline{A}\alpha_2 \mathrm{e}^{-\alpha'\overline{\alpha'}} L(d\alpha').$$

Since $A(\alpha, z)$ is holomorphic in α we gain from (2.6.18) by the Cauchy inequalities

$$|\partial_{\alpha_2} A| = \mathcal{O}\left(\eta^{1/4}\delta^2 h^{-5/2}\right). \tag{2.6.26}$$

Here we used that the first term in (2.6.18) is independent of α. Extend A to a function on $\mathbb{C}^{N-1}$ such that the above estimate still holds. Then, by Lemma 2.6.3 there exists a constant $D > 0$ such that

$$\pi^{-N} B \int\limits_{\|\alpha'\|_{\mathbb{C}^{N-1}} \geq R_0} \overline{A}\alpha_2 \mathrm{e}^{-\alpha'\overline{\alpha'}} L(d\alpha') = \mathcal{O}\left(\eta^{1/2} h^{-1} \delta \mathrm{e}^{-\frac{\asymp \eta^{3/2}}{h}} + \delta^3 h^{-4}\right) \mathrm{e}^{-\frac{D}{h^2}}.$$

Here we used (2.6.18) and (2.6.19). Stokes' theorem and (2.6.26) imply

$$\begin{aligned} \pi^{-N} B \int_{\mathbb{C}^{N-1}} \overline{A}\alpha_2 \mathrm{e}^{-\alpha'\overline{\alpha'}} L(d\alpha') &= \pi^{-N} B \int_{\mathbb{C}^{N-1}} \left(\partial_{\overline{\alpha}_2} \overline{A}\right) \mathrm{e}^{-\alpha'\overline{\alpha'}} L(d\alpha') \\ &\leq \mathcal{O}\left(\delta^3 h^{-3}\right). \end{aligned}$$

Plugging the above into (2.6.25), we gather that there exist a constant $D > 0$ such that

$$\begin{aligned} \pi^{-N} \int\limits_{\|\alpha'\|_{\mathbb{C}^{N-1}} < R_0} & |A - B\alpha_2|^2 \exp\left\{-\alpha'\overline{\alpha'}\right\} L(d\alpha') \\ &= \pi^{-1}\left(|A(z)|^2 + |B(z)|^2\right) + \mathcal{O}\left(\delta^3 h^{-3} + \mathrm{e}^{-\frac{D}{h^2}}\right) \\ &=: \delta^2 \Psi(z, h, \delta). \end{aligned} \tag{2.6.27}$$

By (2.6.18) and (2.6.19), we see that $\pi^{-1}(|A(z)|^2+|B(z)|^2)$ is equal to

$$\begin{aligned}\frac{\delta^2}{\pi}\Big((\partial_z X|\partial_z X) &- \frac{1}{\|X\|^2}|(\partial_z X|X)|^2 \\ &+\delta^{-2}\left|(e_0|f_0)(1+\mathcal{O}\left(h^\infty\right))+\mathcal{O}\left(\eta^{1/4}\delta^2 h^{-7/2}\right)\right|^2\Big).\end{aligned}$$

The above, (2.6.27), (2.6.22) and

$$\left|\delta\|X(z)\|+\mathcal{O}\left(\eta^{-1/4}\delta^2 h^{-3/2}\right)\right|^{-2}=\frac{\left(1+\mathcal{O}\left(\eta^{-1/4}\delta h^{-3/2}\right)\right)}{\delta^2\pi\|X(z)\|^2},$$

imply that for $h>0$ small enough, there exists a constant $D>0$ such that

$$\Lambda_1(0,z):=\frac{\left(1+\mathcal{O}\left(\eta^{-1/4}\delta h^{-3/2}\right)\right)}{\pi\|X(z)\|^2}\mathbb{1}_{\{\sqrt{\Theta(z;h,\delta)}\leq\frac{\tilde{C}}{h}\}}(z)\Psi(z,h,\delta)\exp^{-\Theta(z;h,\delta)}. \tag{2.6.28}$$

Finally, let us estimate Λ_2 from (2.6.21): applying (2.6.18), (2.6.19) and Lemma 2.6.3 to (2.6.17) yields

$$\Lambda_2(0,z)\leq \mathrm{e}^{-\frac{\tilde{C}}{h^2}}\mathcal{O}\left(\delta^4\eta^{1/2}h^{-7}+\eta^{1/2}h^{-1}\delta\mathrm{e}^{-\frac{\asymp\eta^{3/2}}{h}}\right)=\mathcal{O}\left(\mathrm{e}^{-\frac{D}{h^2}}\right),$$

for some $D>0$. Thus, we can substitute $\mathbb{1}_{\{\sqrt{\Theta(z;h,\delta)}\leq\frac{\tilde{C}}{h}\}}(z)$ with 1 in (2.6.28), up to an error of order $\mathcal{O}(\mathrm{e}^{-\frac{D}{h^2}})$.

Step V In this step we will estimate $I_2(\varphi)$ of (2.6.11). Therefore, we increase the space of integration

$$\begin{aligned}&\int\limits_{\substack{B(0,R)\\ \|\alpha'\|_{\mathbb{C}^{N-1}}>R_0}}\chi\left(\frac{E_{-+}^\delta(z,\alpha)}{\varepsilon}\right)\frac{1}{\varepsilon^2}\left|\partial_z E_{-+}^\delta(z,\alpha)\right|^2\mathrm{e}^{-\alpha\overline{\alpha}}L(d\alpha)\\ &\leq\int\limits_{\substack{B(0,2R)\\ R_0<\|\alpha'\|_{\mathbb{C}^{N-1}}<2R_0}}\chi\left(\frac{E_{-+}^\delta(z,\alpha)}{\varepsilon}\right)\frac{1}{\varepsilon^2}\left|\partial_z E_{-+}^\delta(z,\alpha)\right|^2\mathrm{e}^{-\alpha\overline{\alpha}}L(d\alpha)=:W_\varepsilon.\end{aligned}$$

It is easy to see that Lemma 2.5.2 holds true for the set $B(0,2R)\cap\{R_0<\|\alpha'\|_{\mathbb{C}^{N-1}}<2R_0\}$, potentially by choosing a larger $C>0$ in Corollary 1.1.5 larger. We can proceed as in Step IV: perform the same change of variables and the limit of $\varepsilon\to0$. This yields

$$\begin{aligned}\lim_{\varepsilon\to0}W_\varepsilon=&\pi^{-N}\int\limits_{R_0<\|\alpha'\|_{\mathbb{C}^{N-1}}<2R_0}\mathbb{1}_{B(0,2R)}(\alpha_1(0,\alpha',z),\alpha')\left|\partial_{\alpha_1}\beta(\alpha_1,\alpha',z)\right|^{-2}\\ &\cdot|A(\alpha,z)-B(z)\alpha_2|^2\exp\left\{-\alpha'\overline{\alpha'}-\Lambda(z,h,\delta)^2\right\}L(d\alpha').\end{aligned}$$

By (2.6.18), (2.6.19) and Lemma 2.6.3 we see that there exists a constant $D>0$ such that

$$\begin{aligned}&\pi^{-N}\int\limits_{R_0<\|\alpha'\|_{\mathbb{C}^{N-1}}<2R_0}|A-B\alpha_2|^2\mathrm{e}^{-\alpha'\overline{\alpha'}}L(d\alpha')\\ &\leq\mathrm{e}^{-\frac{\tilde{D}}{h^2}}\mathcal{O}\left(\delta^4\eta^{1/2}h^{-7}+\eta^{1/2}h^{-1}\delta\mathrm{e}^{-\frac{\asymp\eta^{3/2}}{h}}\right)=\mathcal{O}\left(\mathrm{e}^{-\frac{D}{h^2}}\right).\end{aligned}$$

The statement about the derivatives of the error terms follows from (2.6.20), (2.6.6). ☐

2.7 | Average Density of Eigenvalues

First, we will give the proof the principal result of Section 1.2:

Proof of Theorem 1.2.12. Due to (1.1.15) and Hypothesis 1.1.6 we have that, for $\kappa > 4$ (as in Hypothesis 1.2.6) large enough, that (1.1.16) holds. Therefore, we assume that $\left(h \ln \frac{1}{h}\right)^{2/3} \ll \eta \leq C$, where $C > 0$ is a constant.

In particular, we now strengthen assumption (2.1.1) and assume from now on that $\Omega \Subset \Sigma$ satisfies Hypothesis 1.1.7 if nothing else is specified, i.e. we assume that

$$\Omega \Subset \Sigma \text{ is open, relatively compact with } \operatorname{dist}(\Omega, \partial\Sigma) \gg \left(h \ln h^{-1}\right)^{2/3}.$$

Recall the definition of $\Omega_\eta^a \cap \Omega$ given in (2.1.2):

$$\Omega_\eta^a = \left\{ z \in \Omega : \frac{\eta}{C} \leq \operatorname{Im} z \leq C\eta \right\}$$

for some constant $C > 0$. Define

$$\widetilde{\Omega}_\eta^a := \left\{ z \in \Omega : \frac{\eta}{2C} \leq \operatorname{Im} z \leq 2C\eta \right\}.$$

Define $\eta_j := C^{-j}$, $j \in \mathbb{N}_0$, and consider the open covering of Ω

$$\Omega \subset \bigcup_{j \in \mathbb{N}_0} \widetilde{\Omega}_{\eta_j}^a \cup \left(\Omega \backslash \bigcup_{j \in \mathbb{N}_0} \overline{\Omega_{\eta_j}^a} \right),$$

where $\operatorname{dist}(\Omega \backslash \bigcup_{j\in\mathbb{N}_0} \overline{\Omega_{\eta_j}^a}, \partial\Sigma) > 1/C$, thus, conforming with the previous notation, we may define

$$\Omega_i := \Omega \backslash \bigcup_{j \in \mathbb{N}_0} \overline{\Omega_{\eta_j}^a}.$$

Let $\{\chi_{\eta_j}\}_{j\in\mathbb{N}_0}$ be a partition of unity subordinate to this locally finite open subcovering such that

$$1 = \sum_{j \in \mathbb{N}} \chi_{\eta_j} + \chi_{\eta_0},$$

in a neighborhood of Ω. Here, for $j \in \mathbb{N}$, $\chi_{\eta_j} \in \mathscr{C}_0^\infty(\widetilde{\Omega}_\eta^a)$, supported in either $\widetilde{\Omega}_\eta^a$. Furthermore, $\chi_{\eta_0} \in \mathscr{C}^\infty(\Omega_i)$. This partition of unity together with Proposition 2.6.1 yields

$$\begin{aligned}
\mathbb{E}\left[\, \Xi(\varphi) \mathbb{1}_{B(0,R)} \,\right] &= \sum_{j\in\mathbb{N}} \mathbb{E}\left[\, \Xi(\varphi\chi_{\eta_j}) \mathbb{1}_{B(0,R)} \,\right] + \mathbb{E}\left[\, \Xi(\varphi\chi_0) \mathbb{1}_{B(0,R)} \,\right] \\
&= \sum_{j\in\mathbb{N}} \int \varphi(z)\chi_{\eta_j}(z) \frac{1 + \mathscr{O}\left(\eta_j^{-1/4}\delta h^{-3/2}\right)}{\pi \|X\|^2} \Psi(z; h, \delta) \mathrm{e}^{-\Theta_j} L(dz) \\
&\quad + \int \varphi(z)\chi_0(z) \frac{1 + \mathscr{O}\left(\delta h^{-3/2}\right)}{\pi \|X\|^2} \Psi(z; h, \delta) \mathrm{e}^{-\Theta_0} L(dz) + \mathscr{O}\left(\mathrm{e}^{-\frac{D}{h^2}}\right).
\end{aligned}$$

where

$$\Theta_j := \frac{\left| E_{-+}(z) + \mathscr{O}\left(\eta_j^{-1/4}\delta^2 h^{-5/2}\right) \right|^2}{\delta^2 \|X\|^2}, \quad \Theta_0 := \frac{|E_{-+}(z) + \mathscr{O}\left(\delta^2 h^{-5/2}\right)|^2}{\delta^2 \|X\|^2}.$$

Note that to gain the exponentially small error estimate in the above we used that the bound on the distribution $T_h \in D'(\mathbb{C})$ (cf. Proposition 2.6.1) is independent of η. Thus,

$$\left| \sum_{j\in\mathbb{N}} \langle T_h, \varphi\chi_{\eta_j} \rangle \right| = |\langle T_h, \varphi \rangle| \leq C \|\varphi\|_\infty \mathrm{e}^{-\frac{D}{h^2}}.$$

Analysis of the density Ψ Recall the formula for the density of eigenvalues given in Proposition 2.6.1. Define

$$\Psi_1(z;h,\delta) := (\partial_z X|\partial_z X) - \frac{1}{\|X\|^2}|(\partial_z X|X)|^2 + \mathcal{O}\left(\delta^3 h^{-3}\right) \tag{2.7.1}$$

Since the error above is of order $\mathcal{O}(1)$, it follows from Proposition 2.4.1 that

$$\Psi_1(z,h,\delta) = \frac{1}{h}\left\{\frac{i}{\{p,\overline{p}\}(\rho_+(z))} - \frac{i}{\{p,\overline{p}\}(\rho_-(z))}\right\} + \mathcal{O}\left(\operatorname{dist}(z,\partial\Sigma)^{-2}\right),$$

where we used that $\operatorname{Im} z \asymp \eta_j$ for $z \in \Omega^a_{\eta_j}$. Proposition 2.4.2 implies

$$\Psi_1(z,h,\delta)L(dz) = \frac{1}{2h}p_*(d\xi\wedge dx) + \mathcal{O}\left(\operatorname{dist}(z,\partial\Sigma)^{-2}\right)L(dz).$$

Furthermore, Proposition 2.6.1 and Proposition 2.4.1 yield that

$$\eta^{-\frac{n+m}{2}}h^{n+m}\partial_z^n\partial_{\overline{z}}^m\mathcal{O}\left(\eta_j^{-2}\right) = \mathcal{O}\left(\eta_j^{-2}\right),$$

where $\mathcal{O}\left(\eta_j^{-2}\right)$ is the error term of Ψ_1. Next, let us turn to the second part of Ψ:

$$\begin{aligned}
&\delta^{-2}\left|(e_0|f_0)(1+\mathcal{O}(h^\infty)) + \mathcal{O}\left(\eta_j^{1/4}\delta^2h^{-7/2}\right)\right|^2\\
&= \delta^{-2}\left|(e_0|f_0)\right|^2(1+\mathcal{O}(h^\infty)) + \mathcal{O}\left(\eta_j^{1/2}\delta^2h^{-7}\right) + \mathcal{O}\left(\eta_j^{1/4}h^{-7/2}\left|(e_0|f_0)\right|\right)\\
&= \delta^{-2}\left|(e_0|f_0)\right|^2(1+\mathcal{O}(h^\infty)) + \mathcal{O}\left(\eta_jh^{-4}\mathrm{e}^{-\frac{S}{h}} + \eta_j^{1/2}\delta^2h^{-7}\right).
\end{aligned}$$

In the last line, we applied an estimate on $|(e_0|f_0)|$ which follows from Proposition 2.2.2 and from Remark 2.2.4. The error term $\mathcal{O}(\eta_jh^{-4}\mathrm{e}^{-\frac{S}{h}})$ is bounded by $\mathcal{O}(\eta_j)$ because $\eta \gg (-h\ln h)^{2/3}$. We then absorb $\mathcal{O}(\eta_j)$ into the error term $\mathcal{O}(\eta_j^{-2})$ of Ψ_1 as well as the error term $\mathcal{O}(\eta_j^{1/2}\delta^2h^{-7}) \leq \mathcal{O}(\eta_j^{1/2})$. Then, one defines

$$\Psi_2(z;h,\delta) := \frac{\left|(e_0|f_0)\right|^2}{\delta^2}\left(1+\mathcal{O}\left(\eta_j^{-3/4}h^{1/2}\right)\right). \tag{2.7.2}$$

As in (2.6.20), the error estimates don't change if we apply $\eta^{-\frac{n+m}{2}}h^{n+m}\partial_z^n\partial_{\overline{z}}^m$.

Analysis of the exponential Θ Recall from Proposition 2.2.1 that $-\alpha_0 = E_{-+}$ and use (2.2.26) to find that

$$E_{-+}(z) = ([P_h,\chi]e_0|f_0)\left(1+\mathcal{O}\left(\exp\left[-\frac{\eta_j^{3/2}}{h}\right]\right)\right).$$

Here $\chi \in \mathscr{C}_0^\infty(S^1)$ with $\chi \equiv 1$ in a small open neighborhood of $\overline{\{x_-(z); z\in\Omega\}}$. Thus, using $\|X\| = (1+\mathcal{O}(h^\infty))$ (cf. Proposition 2.3.3), we have the following equation for Θ given in Proposition 2.6.1

$$\begin{aligned}
\Theta(z,h,\delta) &= \frac{\left|E_{-+}(z) + \mathcal{O}\left(\eta_j^{-1/4}\delta^2h^{-5/2}\right)\right|^2}{\delta^2\|X\|^2}\\
&= \frac{\left|([P_h,\chi]e_0|f_0) + \mathcal{O}\left(\eta_j^{-1/4}\delta^2h^{-5/2}\right)\right|^2}{\delta^2(1+\mathcal{O}(h^\infty))}\left(1+\mathcal{O}\left(\mathrm{e}^{-\frac{\eta_j^{3/2}}{h}}\right)\right).
\end{aligned} \tag{2.7.3}$$

As in (2.6.20), the error estimates stay invariant under the action of $\eta_j^{-\frac{n+m}{2}}h^{n+m}\partial_z^n\partial_{\overline{z}}^m$. Finally, to conclude the density given in the Theorem, note that

$$\frac{1+\mathcal{O}\left(\eta_j^{-1/4}\delta h^{-3/2}\right)}{\pi\|X\|^2} = \frac{1+\mathcal{O}\left(\operatorname{dist}(z,\partial\Sigma)^{-1/4}\delta h^{-3/2}\right)}{\pi}. \qquad \square$$

In the case of the operator P_h^δ, it is possible to state more explicit formulas for the different parts of the density of eigenvalues given in Theorem 1.2.12:

It follows by Propositions 2.4.1 and 2.4.2 that

$$\begin{aligned}\frac{1}{2h}p_*(d\xi\wedge dx) &= \frac{1}{h}\left\{\frac{i}{\{p,\overline{p}\}(\rho_+(z))}+\frac{i}{\{\overline{p},p\}(\rho_-(z))}\right\}L(dz)\\ &\asymp \frac{1}{h\sqrt{\operatorname{dist}(z,\partial\Sigma)}}L(dz)\end{aligned}$$

where we used that $\operatorname{Im} z\asymp\eta_j$ for $z\in\Omega_\eta^a$. For our purposes we can assume that $|\operatorname{Im} z-\langle\operatorname{Im} g\rangle|>1/C$, $C\gg 1$, since inside this tube Ψ_2 and Θ are exponentially small in $h>0$. In the case of Ψ_2, this follows from the assumptions on δ (cf. Hypothesis 1.2.6) and from Remark 2.2.4. In the case of Θ, this follows from the assumptions on δ and Proposition 2.2.12 and (2.7.3). Thus, applying Proposition 2.2.2 to (2.7.2) yields

$$\Psi_2(z;h,\delta)=\frac{\left(\frac{i}{2}\{p,\overline{p}\}(\rho_+)\frac{i}{2}\{\overline{p},p\}(\rho_-)\right)^{\frac{1}{2}}}{\pi h\delta^2}|\partial_{\operatorname{Im} z}S(z)|^2\mathrm{e}^{-\frac{2S}{h}}\left(1+\mathcal{O}\left(\eta^{-3/4}h^{1/2}\right)\right). \tag{2.7.4}$$

As in (2.6.20), the error estimates don't change if we apply $\eta^{-\frac{n+m}{2}}h^{n+m}\partial_z^n\partial_{\overline{z}}^m$. Moreover, since $\operatorname{Im} z\asymp\eta_j$ for $z\in\Omega_\eta^a$,

$$\Psi_2^0(z;h,\delta)\asymp\frac{(\operatorname{dist}(z,\partial\Sigma))^{3/2}\mathrm{e}^{-\frac{2S}{h}}}{h\delta^2}.$$

Apply Proposition 2.2.12 to (2.7.3) gives that

$$\begin{aligned}\Theta(z,h,\delta)=V(z,h)^2\frac{\mathrm{e}^{-\frac{2S}{h}}}{\delta^2}&\left(1+\mathcal{O}\left(h^\infty\right)+\mathcal{O}\left(\mathrm{e}^{-\frac{\asymp\eta_j^{3/2}}{h}}\right)\right)\\ &+\mathcal{O}\left(\eta_j^{-1/2}\delta^2h^{-5}\right)+\mathcal{O}\left(Vh^{-5/2}\mathrm{e}^{-\frac{S}{h}}\right).\end{aligned} \tag{2.7.5}$$

Since $0\le V=\mathcal{O}\left(\eta_j^{1/4}h^{1/2}\right)$ by (2.2.23), it follows that

$$\begin{aligned}\Theta(z,h,\delta)=&\frac{h\left(\frac{i}{2}\{p,\overline{p}\}(\rho_+)\frac{i}{2}\{\overline{p},p\}(\rho_-)\right)^{\frac{1}{2}}}{\pi}\frac{\mathrm{e}^{-\frac{2S}{h}}}{\delta^2}\left(1+\mathcal{O}\left(\eta_j^{-1/4}h^{\frac{3}{2}}\right)\right)\\ &+\mathcal{O}\left(\eta_j^{-1/2}\delta^2h^{-5}\right)+\mathcal{O}\left(\eta_j^{1/4}h^{-2}\mathrm{e}^{-\frac{S}{h}}\right).\end{aligned}$$

Furthermore, for $\mathrm{e}^{-\frac{2S}{h}}\delta^{-2}\le 1$, the error term $\mathcal{O}\left(\eta_j^{1/4}h^{-2}\mathrm{e}^{-\frac{S}{h}}\right)$ is bounded by $\mathcal{O}(\eta_j^{1/4}h^{-2}\delta)$ since there we have that $\mathrm{e}^{-\frac{S}{h}}\le\delta$. For $\mathrm{e}^{-\frac{2S}{h}}\delta^{-2}\le 1$, we have that

$$\mathcal{O}\left(\eta_j^{1/4}h^{-2}\mathrm{e}^{-\frac{S}{h}}\right)\le\mathcal{O}\left(\eta_j^{1/4}h^{-2}\delta\mathrm{e}^{-\frac{2S}{h}}\delta^{-2}\right)\le\mathcal{O}\left(\eta_j^{1/4}h^{2}\mathrm{e}^{-\frac{2S}{h}}\delta^{-2}\right).$$

Thus,

$$\begin{aligned}\Theta(z,h,\delta)=&\frac{h\left(\frac{i}{2}\{p,\overline{p}\}(\rho_+)\frac{i}{2}\{\overline{p},p\}(\rho_-)\right)^{\frac{1}{2}}}{\pi}\frac{\mathrm{e}^{-\frac{2S}{h}}}{\delta^2}\left(1+\mathcal{O}\left(\eta_j^{-1/4}h^{\frac{3}{2}}\right)\right)\\ &+\mathcal{O}\left(\eta_j^{1/4}h^{-2}\delta+\eta_j^{-1/2}\delta^2h^{-5}\right).\end{aligned} \tag{2.7.6}$$

Analogous to (2.6.20), the error estimates stay for $\beta\in\mathbb{N}^2$ invariant under the action of $\eta_j^{-\frac{|\beta|}{2}}h^{|\beta|}\partial_{z\overline{z}}^\beta$. Moreover,

$$\Theta^0(z;h,\delta)\asymp h\sqrt{\operatorname{dist}(z,\partial\Sigma)}\frac{\mathrm{e}^{-\frac{2S}{h}}}{\delta^2}.$$

We have thus proven Proposition 1.2.14 and Proposition 1.2.13. Since we will need it later on we will state the following formulas:

Lemma 2.7.1. *Under the assumptions of Theorem 1.2.12 and for* $\left(h\ln h^{-1}\right)^{2/3} \ll \eta < \text{const}$, *we have*

$$\partial_{\operatorname{Im} z}\Psi_1 = -\frac{1}{4h}\left(\frac{\operatorname{Im} g''(x_-)}{(\operatorname{Im} g'(x_-))^3} - \frac{\operatorname{Im} g''(x_+)}{(\operatorname{Im} g'(x_+))^3}\right) + \mathcal{O}\left(\eta^{-2}\right) = \mathcal{O}\left(\eta^{-3/2}h^{-1}\right)$$

and for $|\operatorname{Im} z - \langle \operatorname{Im} g\rangle| > 1/C$, $C > 0$ *large enough,*

$$\partial_{\operatorname{Im} z}\Psi_2(z,h) = \frac{2\left(\frac{i}{2}\{p,\overline{p}\}(\rho_+)\frac{i}{2}\{\overline{p},p\}(\rho_-)\right)^{\frac{1}{2}}}{\pi h^2}|\partial_{\operatorname{Im} z}S(z)|^2(-\partial_{\operatorname{Im} z}S(z))\frac{\mathrm{e}^{-\frac{2S}{h}}}{\delta^2} \cdot\left(1+\mathcal{O}\left(\eta^{-3/4}h^{\frac{1}{2}}\right)\right),$$

$$\partial_{\operatorname{Im} z}\Theta(z,h) = \frac{2\left(\frac{i}{2}\{p,\overline{p}\}(\rho_+)\frac{i}{2}\{\overline{p},p\}(\rho_-)\right)^{\frac{1}{2}}}{\pi\delta^2\exp\{-\frac{2S}{h}\}}(-\partial_{\operatorname{Im} z}S(z))\left(1+\mathcal{O}\left(\eta^{-1/4}h^{\frac{3}{2}}\right)\right) + \mathcal{O}\left(\eta^{3/4}h^{-3}\delta + \delta^2 h^{-6}\right),$$

Proof. Let us first treat Ψ_1: Recall from the proof of Proposition 2.4.5 that Ψ_1 was given by an oscillatory integral where the phase vanishes at the critical point. Thus, the $\partial_{\operatorname{Im} z}$ derivative of the error term $\mathcal{O}\left(\eta^{-2}\right)$ grows at most by η^{-1}. Thus, taking the derivative of (2.7.1) yields

$$\partial_{\operatorname{Im} z}\Psi_1 = -\frac{1}{4h}\left(\frac{\operatorname{Im} g''(x_-)}{(\operatorname{Im} g'(x_-))^3} - \frac{\operatorname{Im} g''(x_+)}{(\operatorname{Im} g'(x_+))^3}\right) + \mathcal{O}\left(\eta^{-3}\right) = \mathcal{O}\left(\eta^{-3/2}h^{-1}\right),$$

where the last estimate follows from $|2\operatorname{Im} g'(x_\pm| = |\{p,\overline{p}\}(\rho_\pm| \asymp \sqrt{\eta}$ (cf. Proposition 2.4.1) and from the fact that the z- and $\overline{z}$-derivative of the error term grow at most by a factor of $\mathcal{O}(\eta^{1/2}h^{-1})$.

Now let us turn to Ψ_2: one calculates from (2.7.4) that for $|\operatorname{Im} z - \langle \operatorname{Im} g\rangle| > 1/C$

$$\partial_{\operatorname{Im} z}\Psi_2(z,h) = \frac{2\left(\frac{i}{2}\{p,\overline{p}\}(\rho_+)\frac{i}{2}\{\overline{p},p\}(\rho_-)\right)^{\frac{1}{2}}}{\pi h^2}|\partial_{\operatorname{Im} z}S(z)|^2(-\partial_{\operatorname{Im} z}S(z))\frac{\mathrm{e}^{-\frac{2S}{h}}}{\delta^2} \cdot\left(1+\mathcal{O}\left(\eta^{-3/4}h^{\frac{1}{2}}\right)\right).$$

Here we used that the z- and $\overline{z}$-derivative of the error terms grow at most by a factor of $\mathcal{O}(\eta^{1/2}h^{-1})$.

Finally, let us turn to Θ: as in the proof of Proposition 2.2.3 one calculates the formula for $\partial_{\operatorname{Im} z}\Theta$ from (2.7.6). □

2.8 | Properties of the density

In this section we will discuss and prove the results stated in Section 1.2.4.

2.8.1 – Local maximum of the average density

First, we prove the resolvent estimate given in Proposition 1.2.5.

Proof of Proposition 1.2.5. Recall from Section 1.2.2 that the operator $Q(z)$ is self-adjoint and that $|t_0(z)| = |\alpha_0(z)|$. It follows that

$$\|(P_h - z)^{-1}\| = |t_0(z)|^{-1} = |\alpha_0(z)|^{-1}.$$

Recall the Grushin problem posed in Proposition 2.2.1. Since $E_{-+}^{-1} = -\alpha_0$, it follows by Proposition 2.2.12 that

$$\|(P_h - z)^{-1}\| = \frac{\exp\left\{\frac{S}{h}\right\}}{V(z)|1-\mathrm{e}^{\Phi(z)}|\left(1+\mathcal{O}\left(\mathrm{e}^{-\frac{\asymp\eta^{\frac{3}{2}}}{h}}\right)\right)}, \tag{2.8.1}$$

which together with (2.2.23) implies (1.2.3). The result about the asymptotic behavior of the resolvent follows from the above together with the fact that $|\{p,\overline{p}\}(\rho_\pm)| \asymp \sqrt{\eta}$ (cf. Proposition 2.4.1). □

We have split the proof of Proposition 1.2.15 into the following two Lemmata:

Lemma 2.8.1. *Let $z \in \Omega \Subset \Sigma_{c,d}$ with $\Sigma_{c,d}$ as in (1.2.16) and let $S(z)$ be as in Definition 1.2.2. Let $\delta > 0$ and $\varepsilon(h)$ be as in Hypothesis 1.2.6 with $\kappa > 1$ large enough. Moreover, let $E_{-+}(z)$ be as in Proposition 2.2.1. Then,*

- *for $0 < h \ll 1$, there exist numbers $y_\pm(h)$ such that $\varepsilon_0 = S(y_\pm(h))$ with*

$$C^{-1}\left(h\ln h^{-1}\right)^{\frac{2}{3}} \ll y_-(h) < \langle \operatorname{Im} g\rangle - ch\ln h^{-1} < \langle \operatorname{Im} g\rangle + ch\ln h^{-1} < y_+(h) \ll \operatorname{Im} g(b) - C^{-1}\left(h\ln h^{-1}\right)^{\frac{2}{3}},$$

  for some constants $C, c > 1$. Furthermore,

$$y_-(h), (\operatorname{Im} g(b) - y_+(h)) \asymp (\varepsilon_0(h))^{2/3};$$

- *there exists $h_0 > 0$ and a family of smooth curves, indexed by $h \in]h_0, 0[$,*

$$\gamma^h_\pm :]c,d[\longrightarrow \mathbb{C} \text{ with } \operatorname{Re}\gamma^h_\pm(t) = t$$

  such that

$$|E_{-+}(\gamma^h_\pm(t))| = \delta,$$

  and

$$\operatorname{Im}\gamma^h_\pm(t) = y_\pm(\varepsilon_0(h))\left(1 + \mathcal{O}\left(\frac{h}{\varepsilon_0(h)}\right)\right).$$

  Furthermore, there exists a constant $C > 0$ such that

$$\frac{d\operatorname{Im}\gamma^h_\pm}{dt}(t) = \mathcal{O}\left(\exp\left[-\frac{\varepsilon_0(h)}{Ch}\right]\right).$$

Lemma 2.8.2. *Assume the same hypothesis as in Lemma 2.8.1 and let*

$$D(z,h) := \frac{1 + \mathcal{O}\left(\delta h^{-\frac{3}{2}}\operatorname{dist}(z,\partial\Sigma)^{-1/4}\right)}{\pi}\Psi(z;h,\delta)\exp\{-\Theta(z;h,\delta)\}$$

be the average density of eigenvalues of the operator of P_h^δ given in Theorem 1.2.12. Then, there exists $h_0 > 0$ and a family of smooth curves, indexed by $h \in]h_0, 0[$,

$$\Gamma^h_\pm :]c,d[\longrightarrow \mathbb{C},\ \operatorname{Re}\Gamma^h_\pm(t) = t,$$

with $\Gamma_- \subset \{\operatorname{Im} z < \langle \operatorname{Im} g\rangle\}$ and $\Gamma_+ \subset \{\operatorname{Im} z > \langle \operatorname{Im} g\rangle\}$, along which $\operatorname{Im} z \mapsto D(z,h)$ takes its local maxima on the vertical line $\operatorname{Re} z = \text{const.}$ and

$$\frac{d}{dt}\operatorname{Im}\Gamma^h_\pm(t) = \mathcal{O}\left(\frac{h^4}{\varepsilon_0(h)^4}\right).$$

Moreover, for all $c < t < d$

$$|\Gamma^h_\pm(t) - \gamma^h_\pm(t)| \leq \mathcal{O}\left(\frac{h^5}{\varepsilon_0(h)^{13/3}}\right).$$

Proof of Proposition 1.2.15. The first two points of the proposition follow from Lemma 2.8.1 together with the observations that $|E_{-+}(z)| = |\alpha_0| = |t_0(z)|$ (cf. Proposition 2.2.1) and that by (2.8.1)

$$\|(P_h - \gamma^h_\pm)^{-1}\| = \delta^{-1}.$$

The third point has been proven with Lemma 2.8.2. □

Proof of Lemma 2.8.1. Recall from Proposition 1.2.3 that S is strictly monotonous above and below the spectral line, i.e. $\operatorname{Im} z = \langle \operatorname{Im} g\rangle$. Furthermore, recall from Hypothesis 1.2.6 that $-\left(\kappa - \frac{1}{2}\right) h \ln h + Ch \leq \varepsilon_0(h) < S(\langle \operatorname{Im} g\rangle)$. Thus, the implicit function theorem implies that there exist $y_\pm(\varepsilon_0(h)) \in \mathbb{R}$ such that $S(y_\pm(\varepsilon_0(h))) = \varepsilon_0(h)$. Note that in the case where $\varepsilon_0(h)$ is independent of h, the same holds true for $y_\pm(\varepsilon_0)$. For the rest of the proof we will only treat the case where $\operatorname{Im} z \leq \langle \operatorname{Im} g\rangle$ (corresponding to y_-) since the other case is similar.

Consider $z \in \Omega \Subset \Sigma_{c,d}$ with $\operatorname{Re} z = \text{const}$. First, let us prove some a priori estimates: assume that there exists a ζ_- with $h^{2/3} \ll \operatorname{Im} g(a) \leq \zeta_- \leq \langle \operatorname{Im} g\rangle$ such that $|E_{-+}(\operatorname{Re} z + i\zeta_-)|\delta^{-1} = 1$. Recall Proposition 1.2.3 and note that

$$\begin{aligned} S(z) - \varepsilon_0(h) &= \int_{\langle \operatorname{Im} g\rangle}^{\operatorname{Im} z} (\partial_{\operatorname{Im} z} S)(t)\,dt + S(\langle \operatorname{Im} g\rangle) - \varepsilon_0(h) \\ &= \int_{y_-(\varepsilon_0(h))}^{\operatorname{Im} z} (\partial_{\operatorname{Im} z} S)(t)\,dt + S(y_-(\varepsilon_0(h))) - \varepsilon_0(h). \end{aligned} \tag{2.8.2}$$

Recall Proposition 2.2.12 and Hypothesis 1.2.6. It follows by (2.8.2), that if $|\zeta_- - \langle \operatorname{Im} g\rangle| \leq \frac{1}{C}$, $C > 0$ large enough, then $|E_{-+}(\operatorname{Re} z + i\zeta_-)|\delta^{-1} \leq \mathcal{O}\left(\eta^{1/4} \mathrm{e}^{-\frac{1}{Dh}}\right)$ for some $D > 0$ large. Thus, we may assume that, in case it exists,

$$|\zeta_- - \langle \operatorname{Im} g\rangle| > \frac{1}{C}. \tag{2.8.3}$$

We conclude from (2.8.2) that

$$y_-(h) \asymp (\varepsilon_0(h))^{2/3} \tag{2.8.4}$$

and that for $C > 0$ large enough

$$|\langle \operatorname{Im} g\rangle - y_-(\varepsilon)| > \frac{1}{C}. \tag{2.8.5}$$

(2.8.4), (2.8.5) and Hypothesis 1.2.6 imply, for $\kappa > 1$ large enough, the first point of the Lemma.

Now let us prove the existence of the points ζ_-. More precisely, we will prove that for $z \in \Omega\Sigma_{c,d}$ with $\operatorname{Im} z < \langle \operatorname{Im} g\rangle - 1/C$ (cf. (2.8.3)) and fixed $\operatorname{Re} z$ there exist exactly one ζ_- such that

$$|E_{-+}(\operatorname{Re} z, \zeta_-)|\delta^{-1} = 1.$$

For $z \in \Omega \cap \Omega_\eta^a \Subset \Sigma_{c,d}$ one calculates from by Proposition 2.2.12 that

$$\begin{aligned} \partial_{\operatorname{Im} z} |E_{-+}(z)| = \Bigg\{ &-V(z)\frac{\partial_{\operatorname{Im} z} S(z)}{h} |1 - \mathrm{e}^{\Phi(z)}| \left(1 + \mathcal{O}\left(\mathrm{e}^{-\frac{\asymp \eta^{\frac{3}{2}}}{h}}\right)\right) \\ &+ \partial_{\operatorname{Im} z}\left[V(z)|1 - \mathrm{e}^{\Phi(z)}|\left(1 + \mathcal{O}\left(\mathrm{e}^{-\frac{\asymp \eta^{\frac{3}{2}}}{h}}\right)\right)\right]\Bigg\} \mathrm{e}^{-\frac{S(z)}{h}}, \end{aligned} \tag{2.8.6}$$

Recall that V is the product of the normalization factors of the quasimodes e_{wkb} and f_{wkb} when $z \in \Omega$ with $\operatorname{dist}(\Omega, \partial\Sigma) > 1/C$ and the product of the normalization factors of the quasimodes e_{wkb}^η and f_{wkb}^η when $z \in \Omega \cap \Omega_\eta^a$ (cf. (2.2.21)). Since the derivative with respect to $\operatorname{Im} z$ of the imaginary part of their phase function $\operatorname{Im} \phi_\pm$ is equal to zero at $x_\pm$, it follows that

$$|\partial_{\operatorname{Im} z} V(z)| = \mathcal{O}(h^{1/2}\eta^{-3/4}). \tag{2.8.7}$$

The a priori bound (2.8.3) implies that there exists a constant $C > 1$ such that

$$|1 - \mathrm{e}^{\Phi(z)}| = 1 + \mathcal{O}\left(\mathrm{e}^{-\frac{1}{Ch}}\right), \text{ and } \partial_{\operatorname{Im} z}|1 - \mathrm{e}^{\Phi(z)}| = \mathcal{O}\left(\mathrm{e}^{-\frac{1}{Ch}}\right). \tag{2.8.8}$$

The fact that $\partial_{\operatorname{Im} z} S(z) > 0$ (cf. (1.2.3)) implies that $\partial_{\operatorname{Im} z}|E_{-+}(z)| < 0$. Note that in the case where $\operatorname{dist}(\Omega, \partial\Sigma) > 1/C$ one sets in the above $\eta = 1$. Recall from Propositions 2.2.9 and 2.2.10 that V is independent of $\operatorname{Re} z$. Using

$$\partial_{\operatorname{Re} z}|1 - \mathrm{e}^{\Phi(z)}| = \mathcal{O}\left(\mathrm{e}^{-\frac{1}{Ch}}\right),$$

we conclude that

$$\begin{aligned}\partial_{\operatorname{Re} z}|E_{-+}(z)| &= \partial_{\operatorname{Re} z}\left[V(z)|1-\mathrm{e}^{\Phi(z)}|\left(1+\mathcal{O}\left(\mathrm{e}^{-\frac{\asymp\eta^{\frac{3}{2}}}{h}}\right)\right)\right]\mathrm{e}^{-\frac{S(z)}{h}} \\ &= \mathcal{O}\left(\mathrm{e}^{-\frac{\asymp\eta^{\frac{3}{2}}}{h}}\right)\mathrm{e}^{-\frac{S(z)}{h}}. \end{aligned} \tag{2.8.9}$$

This implies that the gradient $|E_{-+}(z)|$ is non-zero for all z with $|\operatorname{Im} z - \langle \operatorname{Im} g\rangle| > 1/C$ (cf. (2.8.3)) and thus we may conclude by the implicit function theorem, that for δ as above there exist locally smooth curves $\gamma_-^h(\operatorname{Re} z) := (\operatorname{Re} z, \zeta_-(\varepsilon_0(h), \operatorname{Re} z)$ such that $|E_{-+}(\gamma_-^h)| = \delta$. Furthermore, we may extend $\gamma_-(\operatorname{Re} z)$ smoothly for $c < \operatorname{Re} z < d$. By the mean value theorem applied to $|E_{-+}(z)|$, there exists a ζ between $y_-(h)$ and $\operatorname{Im}\gamma_-^h(\operatorname{Re} z)$ such that

$$\begin{aligned}&||E_{-+}(\operatorname{Re} z + i y_-(h))| - |E_{-+}(\gamma_-^h(\operatorname{Re} z))|| \\ &= |(\partial_{\operatorname{Im} z}|E_{-+}(z)|)(\operatorname{Re} z + i\zeta)|\cdot|y_-(h) - \operatorname{Im}\gamma_-^h(\operatorname{Re} z)|.\end{aligned}$$

Since $|E_{-+}| = \mathcal{O}(\sqrt{h}\eta^{1/4}\mathrm{e}^{-\frac{S}{h}})$ (cf. Proposition 2.2.6) and $\partial_{\operatorname{Im} z}|E_{-+}| \asymp -h^{-1/2}\eta^{3/4}\mathrm{e}^{-\frac{S}{h}}$ (cf. (2.8.6)), it follows that

$$|y_-(h) - \operatorname{Im}\gamma_-^h(\operatorname{Re} z)| = \mathcal{O}\left(\eta^{-1/2}h\right). \tag{2.8.10}$$

$\eta \asymp y_-(h) \asymp (\varepsilon_0(h))^{2/3}$ implies that also $\operatorname{Im}\gamma_-^h(\operatorname{Re} z) \asymp \eta \asymp (\varepsilon_0(h))^{2/3}$, and we conclude that

$$\operatorname{Im}\gamma_-^h(\operatorname{Re} z) = y_-(\varepsilon_0(h))\left(1+\mathcal{O}\left(\frac{h}{\varepsilon_0(h)}\right)\right).$$

Finally, by

$$\begin{aligned}0 &= \frac{d}{d\operatorname{Re} z}|E_{-+}(\gamma_-^h(\operatorname{Re} z))| \\ &= \partial_{\operatorname{Re} z}|E_{-+}(\gamma_-^h(\operatorname{Re} z))| + \partial_{\operatorname{Im} z}|E_{-+}(\gamma_-^h(\operatorname{Re} z))|\frac{d\operatorname{Im}\gamma_-^h}{d\operatorname{Re} z}(\operatorname{Re} z).\end{aligned}$$

and by (2.8.6) and (2.8.9) we may then conclude

$$\frac{d\operatorname{Im}\gamma_-^h}{d\operatorname{Re} z}(\operatorname{Re} z) = \mathcal{O}\left(\mathrm{e}^{-\frac{\asymp\eta^{3/2}}{h}}\right) \tag{2.8.11}$$

which, using $\eta \asymp y_-(h) \asymp (\varepsilon_0(h))^{2/3}$, yields the last statement of the Lemma. □

Proof of Lemma 2.8.2. The idea of this proof is to search for the critical points of the average density of eigenvalues via the Banach fix point theorem. We shall only consider the case where $\operatorname{Im} z \leq \langle \operatorname{Im} g\rangle$ since the other case is similar.

Recall from Proposition 1.2.13 the explicit form the density given in Theorem 1.2.12. Proposition 2.4.1 and the fact that $\operatorname{Im} g$ has exactly two critical points imply that Ψ_1 is strictly monotonously decreasing. Thus, we may assume similar to (2.8.3) that for $C > 0$ large enough

$$|\operatorname{Im} z - \langle \operatorname{Im} g\rangle| > \frac{1}{C}. \tag{2.8.12}$$

since else $\Psi_2 = \mathcal{O}(\mathrm{e}^{-\frac{1}{Dh}})$ with $D > 0$ large. Now, to find the critical points of the density of eigenvalues consider

$$\begin{aligned}\pi\partial_{\operatorname{Im} z}D(z,h) &= \left(\partial_{\operatorname{Im} z}\Psi(z;h,\delta)\exp\{-\Theta(z;h,\delta)\}\right)\left(1+\mathcal{O}\left(\delta\eta^{-1/4}h^{-3/2}\right)\right) \\ &\quad + \Psi(z;h,\delta)\exp\{-\Theta(z;h,\delta)\}\mathcal{O}\left(\delta\eta^{1/4}h^{-5/2}\right) \\ &= 0.\end{aligned} \tag{2.8.13}$$

Here we used that the z- and $\overline{z}$-derivative of the error term $\mathcal{O}\left(\delta\eta^{-1/4}h^{-3/2}\right)$ increases its order of growth at most by a term of order $\mathcal{O}(\eta^{1/2}h^{-1})$ (cf. Theorem 1.2.12). By

$$\partial_{\operatorname{Im} z}\Psi(z;h,\delta)\mathrm{e}^{-\Theta(z;h,\delta)} = (\partial_{\operatorname{Im} z}\Psi_1 + \partial_{\operatorname{Im} z}\Psi_2 - (\Psi_1+\Psi_2)\partial_{\operatorname{Im} z}\Theta)\,\mathrm{e}^{-\Theta(z;h,\delta)},$$

and by Lemma 2.7.1 and Proposition 1.2.13, we can write (2.8.13) as

$$\begin{aligned}
&h^{-3}F(z,h,\delta) + 2\frac{\mathrm{e}^{-\frac{2S}{h}}}{\delta^2}|\partial_{\operatorname{Im} z}S(z)|^2(-\partial_{\operatorname{Im} z}S(z))\frac{\left(\frac{i}{2}\{p,\overline{p}\}(\rho_+)\frac{i}{2}\{\overline{p},p\}(\rho_-)\right)^{\frac{1}{2}}}{\pi h^2}\\
&\cdot\left(1+\mathcal{O}\left(\eta^{-3/4}h^{1/2}\right)\right)\left((1+\mathcal{O}(\eta^{-3/2}h)) - \frac{h\left(\frac{i}{2}\{p,\overline{p}\}(\rho_+)\frac{i}{2}\{\overline{p},p\}(\rho_-)\right)^{\frac{1}{2}}}{\pi\delta^2\exp\{-\frac{2S}{h}\}}\right)\\
&=0,
\end{aligned}\tag{2.8.14}$$

where $F(z,h,\delta)$ is a function depending smoothly on z, satisfying the bound

$$F(z,h,\delta) \asymp -\frac{h^2}{\eta^{3/2}}.$$

Here we used $\partial_{\operatorname{Im} z}\Psi_1 \asymp -(\eta^{3/2}h)^{-1}$ which follows from Lemma 2.7.1 using the fact that $\operatorname{Im} g$ has only two critical points: a minimum at a and a maximum at b.

Remark 2.8.3. In the case $\operatorname{Im} z > \langle \operatorname{Im} g\rangle$ we find similarly that $F(z,h,\delta) \asymp \frac{h^2}{\eta^{3/2}}$.

Furthermore, the functions in (2.8.14) are smooth in z and the z- and $\overline{z}$-derivative increase their order of growth at most by $\mathcal{O}(\eta^{1/2}h^{-1})$. Recall $|E_- + (z)|$ as given in Proposition 2.2.12 and define

$$l(z) := |E_{-+}(z)| = \frac{h\left(\frac{i}{2}\{p,\overline{p}\}(\rho_+)\frac{i}{2}\{\overline{p},p\}(\rho_-)\right)^{\frac{1}{2}}}{\pi}\frac{\mathrm{e}^{-\frac{2S}{h}}}{\delta^2}(1+\mathcal{O}(\eta^{-3/2}h))$$

Thus, (2.8.14) is equal to zero if and only if

$$G(z,h,\delta) + l(1-l) = 0,\tag{2.8.15}$$

where $G(z,h,\delta)$ is a function depending smoothly on z, satisfying

$$G(z,h,\delta) = \frac{F(z,h,\delta)}{2|\partial_{\operatorname{Im} z}S(z)|^2(-\partial_{\operatorname{Im} z}S(z))}\left(1+\mathcal{O}\left(\eta^{-3/4}h^{1/2}\right)\right) \asymp \frac{h^2}{\eta^3}.$$

The z- and $\overline{z}$-derivative increase the order of growth of G at most by $\mathcal{O}(\eta^{\frac{1}{2}}h^{-1})$. For $l \geq 0$ to be a solution to (2.8.15), it is necessary that

$$l = 1 + \frac{h^2}{\mathcal{O}(1)\eta^3}.$$

Thus, $l \asymp 1$. Define the smooth function

$$z \mapsto t(z) := \frac{\eta^3}{h^2}(l(z)-1),$$

with $-c_0 \leq t \leq C_0$ and $c_0, C_0 > 0$ large enough. As in (2.8.6) on calculates

$$\frac{h^2}{\eta^3}\partial_{\operatorname{Im} z}t = -\frac{2\partial_{\operatorname{Im} z}S}{h}\left(1+\mathcal{O}(\eta^{-3/2}h)\right)l(\operatorname{Im} z) \asymp -\frac{\eta^{1/2}}{h},$$

where we used that $\partial_{\operatorname{Im} z}S \asymp \sqrt{\eta}$ (cf. Proposition 1.2.3) and that $\partial_{\operatorname{Im} z}\left(\frac{i}{2}\{p,\overline{p}\}(\rho_+)\frac{i}{2}\{\overline{p},p\}(\rho_-)\right)^{\frac{1}{2}}$ is of order $\mathcal{O}(\eta^{-1/2})$ due to the scaling $\tilde{z} = z\eta$ as in the proof of Proposition 2.1.11. The implicit function theorem then implies that we may locally invert and that $t \mapsto (\operatorname{Im} z)(t)$ is smooth. Since $-c_0 \leq t \leq$

C_0 we may continue $(\operatorname{Im} z)(t)$ smoothly to all open subsets of the domain of t. Furthermore, we conclude that

$$\frac{d(\operatorname{Im} z)}{dt} \asymp -\eta^{-7/2}h^3 \tag{2.8.16}$$

Substitute $\operatorname{Im} z = \operatorname{Im} z(t)$ in (2.8.15). To find the critical points, it is then enough to consider

$$t - \widetilde{G}(t, \operatorname{Re} z, h, \delta) = 0, \quad \widetilde{G}(t, \operatorname{Re} z, h, \delta) := \frac{G(\operatorname{Im} z(t), \operatorname{Re} z, h, \delta))}{\eta^{-3}h^2(1+\eta^{-3}h^2 t)}$$

and one finds

$$\frac{d}{dt}\widetilde{G}(t, \operatorname{Re} z, h, \delta)) = \mathcal{O}(h^2\eta^{-3}).$$

Thus, using $t(\gamma_-^h) = 0$ as starting point, which corresponds to $l(\gamma_-^h) = 1$, the Banach fixed-point theorem implies that for each $\operatorname{Re} z$ there exist a unique zero, $t_-^*(\operatorname{Re} z)$, of (2.8.14), it depends smoothly on $\operatorname{Re} z$ and satisfies

$$|t_-^*(\operatorname{Re} z) - t(\gamma_-^h)| \leq \mathcal{O}(h^2\eta^{-3}). \tag{2.8.17}$$

and

$$\begin{aligned}\frac{dt_-^*(\operatorname{Re} z)}{d\operatorname{Re} z} &= \frac{1}{1-\left(\frac{d}{dt}\widetilde{G}\right)(t_-^*, \operatorname{Re} z, h, \delta)}(\partial_{\operatorname{Re} z}\widetilde{G})(t_-^*, \operatorname{Re} z, h, \delta)) \\ &= \frac{1}{1+\mathcal{O}(h^2\eta^{-3})}(\partial_{\operatorname{Re} z}\widetilde{G})(t_-^*, \operatorname{Re} z, h, \delta)).\end{aligned}$$

Since the z- and $\overline{z}$-derivative applied to G increase its order of growth at most by $\mathcal{O}(\eta^{1/2}h^{-1})$, we conclude that

$$\frac{dt_-^*(\operatorname{Re} z)}{d\operatorname{Re} z} = \mathcal{O}(\eta^{1/2}h^{-1}).$$

Taylor's formula applied to $(\operatorname{Im} z)(t)$ yields that

$$(\operatorname{Im} z)(t_\pm^*(\operatorname{Re} z)) = \operatorname{Im} z(t(\operatorname{Im}\gamma_\pm^h(\operatorname{Re} z))) + \int_{t(\operatorname{Im}\gamma_\pm^h(\operatorname{Re} z))}^{t_\pm^*(\operatorname{Re} z)} \frac{d\operatorname{Im} z}{dt}(\tau)d\tau.$$

By (2.8.17) and (2.8.16) we conclude that

$$(\operatorname{Im} z)(t_\pm^*(\operatorname{Re} z)) = \operatorname{Im}\gamma_\pm^h(\operatorname{Re} z) + \mathcal{O}(\eta^{-13/2}h^5) \tag{2.8.18}$$

and using (2.8.11) that

$$\frac{d}{d\operatorname{Re} z}(\operatorname{Im} z)(t_\pm^*(\operatorname{Re} z)) = \mathcal{O}\left(\eta^{-6}h^4\right).$$

It follows by Proposition 1.2.17 that the density has local maxima along the curves $\Gamma_\pm^h(\operatorname{Re} z) := (\operatorname{Re} z, \operatorname{Im} z(t_\pm^*(\operatorname{Re} z)))$. Applying this definition to (2.8.18) yields that

$$|\operatorname{Im}\Gamma_\pm^h(\operatorname{Re} z) - \operatorname{Im}\gamma_\pm^h(\operatorname{Re} z)| \leq \mathcal{O}(\eta^{-13/2}h^5)$$

for all $z \in \Sigma_{c,d}$. By Lemma 2.8.1 we have that $\operatorname{Im}\gamma_\pm^h(\operatorname{Re} z) \asymp \varepsilon_0(h)^{2/3}$. Thus,

$$\operatorname{Im}\Gamma_\pm^h(\operatorname{Re} z) = \operatorname{Im}\gamma_\pm^h(\operatorname{Re} z)\left(1+\mathcal{O}(\varepsilon_0(h)^{-5}h^5)\right),$$

which in particular implies that $\operatorname{Im}\Gamma_\pm^h(\operatorname{Re} z) \asymp \varepsilon_0(h)^{2/3}$. This concludes the proof of the lemma. □

Proof of Proposition 1.2.17. Proposition 1.2.13 implies that for $|\operatorname{Im} z - \langle \operatorname{Im} g\rangle| > 1/C$

$$\Psi_2(z,h,\delta) = \frac{\left(\frac{i}{2}\{p,\overline{p}\}(\rho_+)\frac{i}{2}\{\overline{p},p\}(\rho_-)\right)^{\frac{1}{2}}}{\pi h\delta^2}e^{-\frac{2S}{h}}|\partial_{\operatorname{Im} z}S(z)|^2\left(1+\mathcal{O}\left(\eta^{-3/4}h^{1/2}\right)\right) \tag{2.8.19}$$

and

$$\Theta(z;h,\delta) = \frac{h\left(\frac{i}{2}\{p,\overline{p}\}(\rho_+)\frac{i}{2}\{\overline{p},p\}(\rho_-)\right)^{\frac{1}{2}}}{\pi}\frac{\mathrm{e}^{-\frac{2S}{h}}}{\delta^2}\left(1+\mathcal{O}\left(\eta^{-1/4}h^{\frac{3}{2}}\right)\right) \\ +\mathcal{O}\left(\eta^{1/4}h^{-2}\delta+\eta^{-1/2}\delta^2h^{-5}\right).$$

Thus, one calculates

$$\left|\Psi_2 - \frac{|\partial_{\operatorname{Im} z}S|^2}{h^2}\Theta\right| \leq \frac{\left(\frac{i}{2}\{p,\overline{p}\}(\rho_+)\frac{i}{2}\{\overline{p},p\}(\rho_-)\right)^{\frac{1}{2}}}{\pi h\delta^2}\mathrm{e}^{-\frac{2S}{h}}|\partial_{\operatorname{Im} z}S(z)|^2\mathcal{O}\left(\eta^{-\frac{3}{4}}h^{\frac{1}{2}}\right) \\ +\mathcal{O}\left(\eta^{5/4}h^{-4}\delta+\eta^{1/2}\delta^2h^{-7}\right),$$

which implies the result given in Proposition 1.2.5. □

Proof of Proposition 1.2.16. We will only consider the case $z \in \Sigma_{c,d}$ with $\operatorname{Im} z \leq \langle \operatorname{Im} g\rangle$ since the case of $\operatorname{Im} z > \langle \operatorname{Im} g\rangle$ is similar.

A priori restrictions on the domain of integration Let $y_-(h)$ and $\gamma_-(\operatorname{Re} z)$ be as in Lemma 2.8.1 and note that similarly to (2.8.2), we have

$$S(\operatorname{Im} z) - \varepsilon_0(h) = \int_{y_-(h)}^{\operatorname{Im}\gamma_-^h}(\partial_{\operatorname{Im} z}S)(t)\,dt + \int_{\operatorname{Im}\gamma_-^h}^{\operatorname{Im} z}(\partial_{\operatorname{Im} z}S)(t)\,dt. \tag{2.8.20}$$

Recall from (2.8.10) that $(\operatorname{Im}\gamma_-^h - y_-(h)) = \mathcal{O}(h\eta^{-1/2})$. Then, one calculates using the mean value theorem and Proposition 1.2.3, similar as in the proof of Lemma 2.8.1 (cf. (2.8.4)), that

$$\int_{y_-(h)}^{\operatorname{Im}\gamma_-^h}(\partial_{\operatorname{Im} z}S)(t)\,dt = \mathcal{O}(h).$$

and that

$$\int_{\operatorname{Im}\gamma_-^h}^{\operatorname{Im} z}(\partial_{\operatorname{Im} z}S)(t)\,dt \asymp (\operatorname{Im} z - \operatorname{Im}\gamma_-^h)\eta^{1/2},$$

where η should be set to 1 in case of $\operatorname{dist}(z,\partial\Sigma_{c,d}) > 1/C$. Next, (2.8.20) and Proposition 1.2.13 imply that

$$\Theta(z;h,\delta) = \frac{\eta^{1/2}}{\mathcal{O}(1)}\exp\left\{-\asymp\frac{(\operatorname{Im} z - \operatorname{Im}\gamma_-^h)\eta^{1/2}}{h}\right\} \\ +\mathcal{O}\left(\eta^{1/4}h^{-2}\delta+\eta^{-1/2}\delta^2h^{-5}\right).$$

Here, we used that $\delta = \sqrt{h}\exp\{-\frac{\varepsilon_0(h)}{h}\}$; see Hypothesis 1.2.6. Thus, for $\operatorname{Im}\gamma_-^h < \operatorname{Im} z < \langle \operatorname{Im} g\rangle$

$$\exp\{-\Theta(z;h,\delta)\} = \left(1+\mathcal{O}\left(\eta^{1/2}\exp\left\{-\frac{(\operatorname{Im} z - \operatorname{Im}\gamma_-^h)\eta^{1/2}}{Ch}\right\}+\eta^{1/4}h^2\right)\right) \tag{2.8.21}$$

and for $\operatorname{Im} z \leq \operatorname{Im}\gamma_-^h$

$$\frac{1}{C}\exp\left\{-C\eta^{1/2}\exp\left[-\frac{(\operatorname{Im} z - \operatorname{Im}\gamma_-^h)\eta^{1/2}}{Ch}\right]\right\} \leq \exp\{-\Theta(z;h,\delta)\} \\ \leq C\exp\left\{-\frac{\eta^{1/2}}{C}\exp\left[-\frac{C(\operatorname{Im} z - \operatorname{Im}\gamma_-^h)\eta^{1/2}}{h}\right]\right\}. \tag{2.8.22}$$

Similarly, by Proposition 1.2.13

$$\Psi_2(z;h,\delta) \le \frac{\eta^{3/2}}{\mathcal{O}(1)h^2}\left(1+\mathcal{O}(\eta^{-1})e^{\Phi(z,h)}\right)\exp\left\{-\frac{(\operatorname{Im} z - \operatorname{Im}\gamma_-^h)\eta^{1/2}}{Ch}\right\}.$$

Thus, for $\operatorname{Im}\gamma_-(\operatorname{Re} z) + \alpha h\eta^{-1/2}\ln\frac{\eta^{1/2}}{h} \le \operatorname{Im} z \le \langle \operatorname{Im} g\rangle$ with $\alpha > 0$ large enough, we see that the average density of eigenvalues (cf. Theorem 1.2.12)

$$D(z,h,\delta)L(dz) = \frac{1}{2h}p_*(d\xi\wedge dx) + \mathcal{O}(\eta^{-2})L(dz). \tag{2.8.23}$$

We then conclude the first statement of the proposition.

Next, recall from Corollary 1.1.5 that restricting the probability space to the ball $B(0,R)$ of radius $R = Ch^{-1}$ implies that $\|Q_\omega\| \le C/h$ with probability $\ge \left(1-e^{-\frac{1}{Ch^2}}\right)$. It follows from

$$\|(P_h^\delta - z)^{-1}\| = \left\|(P_h-z)^{-1}\sum_{n\ge 1}(-\delta)^n\left(Q_\omega(P_h-z)^{-1}\right)^n\right\|$$

that for $z\notin\sigma(P_h)$ such that $\delta\|Q_\omega\|\|(P_h-z)^{-1}\| < 1$, we have that $z\notin\sigma(P_h^\delta)$ with probability $\ge \left(1-e^{-\frac{1}{Ch^2}}\right)$. Proposition 1.2.5 implies that with probability $\ge\left(1-e^{-\frac{1}{Ch^2}}\right)$

$$\delta\|Q_\omega\|\|(P_h-z)^{-1}\| \le \frac{C\left|1-e^{\Phi(z,h)}\right|^{-1}}{h^{3/2}\left(\frac{i}{2}\{p,\overline{p}\}(\rho_+)\frac{i}{2}\{\overline{p},p\}(\rho_-)\right)^{\frac{1}{4}}}\exp\left\{\frac{S(z)-\varepsilon_0(h)}{h}\right\}.$$

Here we used as well Hypothesis 1.2.6. Since $S(z)\asymp\eta^{3/2}$, it follows that $\eta\asymp\varepsilon_0(h)^{2/3}$. Using the mean value theorem together with Proposition 2.8.1 implies that with probability $\ge\left(1-e^{-\frac{1}{Ch^2}}\right)$ there are no eigenvalues of P_h^δ with

$$\operatorname{Im} z \le \beta_1 := \operatorname{Im}\gamma_-^h - C\frac{h}{\varepsilon_0(h)^{1/3}}\ln\left(\frac{\varepsilon_0(h)^{1/6}}{h}\right),\quad C\gg 1.$$

Thus, to count eigenvalues it is sufficient to integrate the density given in Theorem 1.2.12 over subsets of

$$\Sigma'_{c,d} = \{z\in\Sigma_{c,d} \mid \beta_1\le\operatorname{Im} z\le\langle\operatorname{Im} g\rangle,\ c<\operatorname{Re} z<d\}.$$

Similarly, for an α large enough as above, define

$$\alpha_1 := \operatorname{Im}\gamma_-(\operatorname{Re} z) + \alpha\frac{h}{\varepsilon_0^{1/3}}\ln\frac{\varepsilon_0(h)^{1/3}}{h}$$

and note that (2.8.23) implies the second statement of the proposition for $\operatorname{Im} z\ge\alpha_1$.

Approximate Primitive Define $d(z) := \operatorname{dist}(z,\partial\Sigma)$ and recall from (2.1.2) that $\eta\asymp d(z)$. Recall that the density of eigenvalues given in Theorem 1.2.12 is given by Ψ_1, Ψ_2 and Θ which are expressed explicitly in Proposition 1.2.13 and Theorem 1.2.12. Since $\operatorname{Im} g(x_\pm) = \operatorname{Im} z$ and $\xi_\pm = \operatorname{Re} z - \operatorname{Re} g(x_\pm)$ (cf (1.1.14)), we conclude together with Proposition 2.4.2 that for $\beta_1\le\operatorname{Im} z\le\alpha_1$

$$\Psi_1(z;h) = \frac{1}{2h}\partial_{\operatorname{Im} z}(x_-(z)-x_+(z)) + \mathcal{O}(d(z)^{-2}) = \frac{1}{2h}\partial^2_{\operatorname{Im} z}S(z) + \mathcal{O}(d(z)^{-2}).$$

Next, it follows by (2.8.19) and Lemma 2.7.1 that

$$|2h\Psi_2 - (\partial_{\operatorname{Im} z}S)(-\partial_{\operatorname{Im} z}\Theta)| = \mathcal{O}\left(d(z)^{3/4}h^{1/2}\frac{e^{-\frac{2S}{h}}}{\delta^2}\right) + \mathcal{O}(d(z)^{3/4}h^{-3}\delta).$$

Thus,

$$\begin{aligned}&\frac{1+\mathcal{O}\big(\delta d(z)^{-1/4}h^{-3/2}\big)}{\pi}\{\Psi_1(z;h)+\Psi_2(z;h,\delta)\}\,\mathrm{e}^{-\Theta(z;h,\delta)}\\&=\frac{1}{2\pi h}\partial_{\operatorname{Im} z}\left[(\partial_{\operatorname{Im} z}S(z))\mathrm{e}^{-\Theta(z;h,\delta)}\right]+R(z;h,\delta)\mathrm{e}^{-\Theta(z;h,\delta)},\end{aligned}\tag{2.8.24}$$

where

$$R(z;h,\delta):=\mathcal{O}\left(d(z)^{-2}+d(z)^{3/4}h^{-1/2}\frac{\mathrm{e}^{-\frac{2S}{h}}}{\delta^2}\right).$$

Let $\beta_1\le\beta_2\le\alpha_1$. Let us first treat the error term R. Similar as for (2.8.21), it follows that

$$R(z;h,\delta)=\mathcal{O}\left(d(z)^{-2}+d(z)^{-3/4}h^{-1/2}\exp\left\{-\frac{(\operatorname{Im} z-\operatorname{Im}\gamma_-^h)d(z)^{1/2}}{Ch}\right\}\right).$$

Hence,

$$\begin{aligned}\left|\int_{\beta_1}^{\alpha_1}R(z;h,\delta)\mathrm{e}^{-\Theta(z;h,\delta)}d(\operatorname{Im} z)\right|&\le\frac{[d(z)^{-1}]_{\beta_1}^{\alpha_1}}{\mathcal{O}(1)}\exp\{-\Theta(\operatorname{Re} z,\alpha_1;h,\delta)\}\\&\quad+\frac{d(z)^{1/4}h^{1/2}}{\mathcal{O}(1)}\exp\left[-\exp\left\{-\frac{(\operatorname{Im} z-\operatorname{Im}\gamma_-^h)d(z)^{1/2}}{Ch}\right\}\right]\Bigg|_{\beta_1}^{\alpha_1}\\&=\frac{\beta_1^{-1}}{\mathcal{O}(1)}\exp\left[-\exp\left\{-\frac{(\alpha_1-\operatorname{Im}\gamma_-^h)\alpha_1^{1/2}}{Ch}\right\}\right]\\&=\frac{\varepsilon_0(h)^{-2/3}}{\mathcal{O}(1)}.\end{aligned}\tag{2.8.25}$$

Next,

$$\begin{aligned}&\frac{1}{2\pi h}\int_{\beta_2}^{\alpha_1}\partial_{\operatorname{Im} z}\big[(\partial_{\operatorname{Im} z}S(z))\mathrm{e}^{-\Theta(z;h,\delta)}\big]L(\operatorname{Im} z)\\&\qquad=\frac{1}{2\pi h}(x_-(\operatorname{Im} z)-x_+(\operatorname{Im} z))\mathrm{e}^{-\Theta(z;h,\delta)}\Big|_{\beta_2}^{\alpha_1}.\end{aligned}\tag{2.8.26}$$

Since,

$$\int\limits_{\substack{\Sigma_{c,d}\\0\le\operatorname{Im} z\le\alpha_1}}\frac{1}{2\pi h}p_*(d\xi\wedge dx)(dz)=\frac{1}{2\pi h}(x_-(\alpha_1)-x_+(\alpha_1))\int_c^d d\operatorname{Re} z$$

we conclude by (2.8.22) the second statement of the proposition for

$$\beta_2=\operatorname{Im}\gamma_-(\operatorname{Re} z)-\frac{h}{\varepsilon_0(h)^{1/3}}\ln\left(\beta\ln\frac{\varepsilon_0(h)^{1/3}}{h}\right)$$

with $\beta>0$ large enough. The last statement of the proposition can be deduced similarly from (2.8.22), (2.8.26) and (2.8.25). □

CHAPTER 3

EIGENVALUE INTERACTION FOR A CLASS OF NON-SELF-ADJOINT OPERATORS UNDER RANDOM PERTURBATIONS

The objective of this section is to build on the results obtained in Chapter 2 and to prove the results discussed in Section 1.3. We consider Hager's model operator P_h (cf (1.1.9)) subject to random perturbations with a small coupling constant δ. We study the 2-point intensity measure of the random point process of eigenvalues of the randomly perturbed operator P_h^δ and prove an h-asymptotic formula for the average 2-point density of eigenvalues. With this we show that two eigenvalues of P_h^δ in the interior of Σ exhibit close range repulsion and long range decoupling. The results presented in this chapter can be found in [82].

3.1 | A formula for the two-point intensity measure

In this section we will give a short reminder of a well-posed Grushin problem for the perturbed operator P_h^δ which has already been used in Chapter 2 (see also [67, 32]). We will then employ the resulting effective Hamiltonians to derive a formula for the two-point intensity measure defined in (1.3.3).

We recall that we always suppose that $\Omega \Subset \mathring{\Sigma}$ is such that Hypothesis 1.3.1 is satisfied, if nothing else is specified.

A Grushin Problem for the perturbed operator P_h^δ As was discussed in Chapter 2, we us the eigenfunctions of the operators Q and $\widetilde{Q}$ (cf (1.2.4)) to create a well-posed Grushin Problem.

Proposition 3.1.1. *Let $z \in \Omega \Subset \Sigma$ with $\mathrm{dist}(\Omega, \partial\Sigma) > 1/C$ and let α_0, e_0 and f_0 be as in (1.2.8). Define*

$$R_+ : H^1(S^1) \longrightarrow \mathbb{C} : u \longmapsto (u|e_0)$$
$$R_- : \mathbb{C} \longrightarrow L^2(S^1) : u_- \longmapsto u_- f_0.$$

Then

$$\mathscr{P}(z) := \begin{pmatrix} P_h - z & R_- \\ R_+ & 0 \end{pmatrix} : H^1(S^1) \times \mathbb{C} \longrightarrow L^2(S^1) \times \mathbb{C}$$

is bijective with the bounded inverse

$$\mathscr{E}(z) = \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{pmatrix}$$

where $E_-(z)v = (v|f_0)$, $E_+(z)v_+ = v_+e_0$, $E(z) = (P_h - z)^{-1}|_{(f_0)^\perp \to (e_0)^\perp}$ *and* $E_{-+}(z)v_+ = -\alpha_0 v_+$. *Furthermore, we have the estimates for* $z \in \Omega$

$$\begin{aligned}
&\|E_-(z)\|_{L^2\to\mathbb{C}}, \|E_+(z)\|_{\mathbb{C}\to H^1} = \mathscr{O}(1),\\
&\|E(z)\|_{L^2\to H^1} = \mathscr{O}(h^{-1/2}),\\
&|E_{-+}(z)| = \mathscr{O}\left(\sqrt{h}\mathrm{e}^{-\frac{S}{h}}\right) = \mathscr{O}\left(\mathrm{e}^{-\frac{1}{Ch}}\right);
\end{aligned} \tag{3.1.1}$$

Definition 3.1.2. For $x \in \mathbb{R}$ we denote the integer part of x by $\lfloor x \rfloor$. Let $C_1 > 0$ be big enough as above and define $N := (2\lfloor \frac{C_1}{h} \rfloor + 1)^2$. Let e_0 and f_0 be as in (1.2.8), let $z \in \Omega \Subset \Sigma$ and let $\widehat{e_0}(z;\cdot)$ and $\widehat{f_0}(z;\cdot)$ denote the Fourier coefficients of e_0 and f_0. We define the vector $X(z) = (X_{j,k}(z))_{|j|,|k|\le\lfloor\frac{C_1}{h}\rfloor} \in \mathbb{C}^N$ to be given by

$$X_{j,k}(z) = \widehat{e_0}(z;k)\overline{\widehat{f_0}(z;j)}, \quad \text{for } |j|,|k| \le \left\lfloor \frac{C_1}{h} \right\rfloor.$$

Proposition 3.1.3. *Let* $z \in \Omega \Subset \Sigma$. *Let* N *be as in Definition 3.1.2 and let* $B(0,R) \subset \mathbb{C}^N$ *be the ball of radius* $R := C/h$, $C > 0$ *large, centered at* 0. *Let* R_-, R_+ *be as in Proposition 3.1.1. Then*

$$\mathscr{P}_\delta(z) := \begin{pmatrix} P_h^\delta - z & R_- \\ R_+ & 0 \end{pmatrix} : H^1(S^1) \times \mathbb{C} \longrightarrow L^2(S^1) \times \mathbb{C}$$

is bijective with the bounded inverse

$$\mathscr{E}_\delta(z) = \begin{pmatrix} E^\delta(z) & E_+^\delta(z) \\ E_-^\delta(z) & E_{-+}^\delta(z) \end{pmatrix}$$

where

$$\begin{aligned}
E^\delta(z) &= E(z) + \mathscr{O}\left(\delta h^{-2}\right) = \mathscr{O}(h^{-1/2})\\
E_-^\delta(z) &= E_-(z) + \mathscr{O}\left(\delta h^{-3/2}\right) = \mathscr{O}(1)\\
E_+^\delta(z) &= E_+(z) + \mathscr{O}\left(\delta h^{-3/2}\right) = \mathscr{O}(1)
\end{aligned}$$

and

$$E_{-+}^\delta(z) = E_{-+}(z) - \delta X(z)\cdot\alpha + T(z;\alpha), \tag{3.1.2}$$

with $X(z)\cdot\alpha = E_-Q_\omega E_+$, $\alpha \in B(0,R)$, *and*

$$T(z,\alpha) := \sum_{n=1}^{\infty} (-\delta)^{n+1} E_- Q_\omega (EQ_\omega)^n E_+ = \mathscr{O}(\delta^2 h^{-5/2}). \tag{3.1.3}$$

Here, the dot-product $X(z)\cdot\alpha$ *is the natural bilinear one.*

Remark 3.1.4. The effective Hamiltonian $E_{-+}^\delta(z)$ depends smoothly on $z \in \Omega$ and holomorphically on $\alpha \in B(0,R) \subset \mathbb{C}^N$. As in (2.6.6) and Proposition 2.2.6, we have the following estimates: for all $z \in \Omega$, all $\alpha \in B(0,R)$ and all $\beta \in \mathbb{N}^2$

$$\begin{aligned}
&\partial_{z\bar{z}}^\beta E_{-+}(z) = \mathscr{O}\left(h^{-|\beta|+1/2}\mathrm{e}^{-\frac{S}{h}}\right), \text{ and}\\
&\partial_{z\bar{z}}^\beta T(z,\alpha) = \mathscr{O}\left(\delta^2 h^{-(|\beta|+\frac{5}{2})}\right)
\end{aligned}$$

where S is as in Definition 1.2.2.

Moreover, as remarked in [67] the effective Hamiltonian $E_{-+}^\delta(z)$ satisfies a $\overline{\partial}$-equation, i.e. there exists a smooth function $f^\delta : \Omega \to \mathbb{C}$ such that

$$\partial_{\bar{z}} E_{-+}^\delta(z) + f^\delta(z) E_{-+}^\delta(z) = 0.$$

This implies that the zeros of $E_{-+}^\delta(z)$ are isolated and countable and we may use the same notion of multiplicity as for holomorphic functions.

3.1.1 – Counting zeros

By the above well-posed Grushin Problem for the perturbed operator P_h^δ we have that $\sigma(P_h^\delta) = (E_{-+}^\delta)^{-1}(0)$. Hence, to study the the two-point intensity measure ν defined in (1.3.3), we investigate the integral

$$\pi^{-N}\int_{B(0,R)}\Bigg(\sum_{\substack{z,w\in(E_{-+}^\delta)^{-1}(0)\\ z\neq w}}\varphi(z,w)\Bigg)\mathrm{e}^{-\alpha^*\cdot\alpha}L(d\alpha) = \int_{\mathbb{C}^2}\varphi(z_1,z_2)d\nu(z_1,z_2)$$

with $\varphi\in\mathscr{C}_0(\Omega\times\Omega)$. Using Remark 3.1.4, we see that the integral is finite since the number of pairs of zeros of $E_{-+}^\delta(\cdot,\alpha)$ in $\operatorname{supp}\varphi$ is uniformly bounded for $\alpha\in B(0,R)$.

Recall the definition of the point process Ξ given in (1.2.10). Using Lemma 2.5.1, we get the following regularization of the 2-fold counting measure $\Xi\otimes\Xi$

$$\langle\varphi,\Xi\otimes\Xi\rangle = \lim_{\varepsilon\to 0^+}\iint\varphi(z_1,z_2)\prod_{j=1}^{2}\varepsilon^{-2}\chi\left(\frac{E_{-+}^\delta(z_l)}{\varepsilon}\right)|\partial_{z_l}E_{-+}^\delta(z_l)|^2L(dz_1)L(dz_2),$$

where $\chi\in\mathscr{C}_0^\infty(\mathbb{C})$ such that $\int\chi(w)L(dw)=1$. Assuming that $\varphi\in\mathscr{C}_0(\Omega\times\Omega)$ is such that $\{(z,z);\ z\in\Omega\}\not\subset\operatorname{supp}\varphi$, we see by the Lebesgue dominated convergence theorem that the two-point intensity measure of the point process Ξ is given by

$$\int_{\mathbb{C}^2}\varphi(z_1,z_2)d\nu(z_1,z_2) = \lim_{\varepsilon\to 0^+}\iint\varphi(z_1,z_2)K_\varepsilon^\delta(z_1,z_2;h)L(dz_1)L(dz_2) \tag{3.1.4}$$

with

$$K_\varepsilon^\delta(z_1,z_2;h) := \int_{B(0,R)}\left[\prod_{l=1}^{2}\varepsilon^{-2}\chi\left(\frac{E_{-+}^\delta(z_l)}{\varepsilon}\right)|\partial_{z_l}E_{-+}^\delta(z_l)|^2\right]\mathrm{e}^{-\alpha^*\alpha}L(d\alpha).$$

Using (3.1.2), we see that the main object of interest is the random vector

$$\begin{aligned} F^\delta(z,w,\alpha;h) &= \begin{pmatrix} E_{-+}^\delta(z)\\ E_{-+}^\delta(w)\\ (\partial_z E_{-+}^\delta)(z)\\ (\partial_z E_{-+}^\delta)(w)\end{pmatrix} \\ &= \begin{pmatrix} E_{-+}(z)\\ E_{-+}(w)\\ (\partial_z E_{-+})(z)\\ (\partial_z E_{-+})(w)\end{pmatrix} - \delta\begin{pmatrix} X(z)\cdot\alpha\\ X(w)\cdot\alpha\\ (\partial_z X)(z)\cdot\alpha\\ (\partial_z X)(w)\cdot\alpha\end{pmatrix} + \begin{pmatrix} T(z,\alpha)\\ T(w,\alpha)\\ (\partial_z T)(z,\alpha)\\ (\partial_z T)(w,\alpha)\end{pmatrix}. \end{aligned} \tag{3.1.5}$$

It will be very useful in the sequel to define the following Gramian matrix G.

$$G := \begin{pmatrix} A & B\\ B^* & C\end{pmatrix}\in\mathbb{C}^{4\times 4}, \tag{3.1.6}$$

with

$$\begin{aligned} A &:= \begin{pmatrix} (X(z)|X(z)) & (X(z)|X(w))\\ (X(w)|X(z)) & (X(w)|X(w))\end{pmatrix},\\ B &:= \begin{pmatrix} (X(z)|\partial_z X(z)) & (X(z)|\partial_w X(w))\\ (X(w)|\partial_z X(z)) & (X(w)|\partial_w X(w))\end{pmatrix},\\ C &:= \begin{pmatrix} (\partial_z X(z)|\partial_z X(z)) & (\partial_z X(z)|\partial_w X(w))\\ (\partial_w X(w)|\partial_z X(z)) & (\partial_w X(w)|\partial_w X(w))\end{pmatrix}. \end{aligned} \tag{3.1.7}$$

Notice that the matrices A,B,C depend on h; see Definition 3.1.2. Next, we will state a formula for the Lebesgue density of the two-point intensity measure ν in terms of the permanent of the Shur complement of G, i.e $\Gamma := C - B^*A^{-1}B$. The permanent of a matrix is defined as follows (cf. [47]):

Definition 3.1.5. Let $(M_{ij})_{ij} = M \in \mathbb{C}^{n\times n}$ be a square matrix and let S_n denote the symmetric group of order n. The permanent of M is defined by

$$\operatorname{perm} M := \sum_{\sigma\in S_n} \prod_{i=1}^{n} M_{i\sigma(i)}. \tag{3.1.8}$$

Remark 3.1.6. Although the definition of the permanent resembles closely to that of the determinant, the two object are quite different. Many properties known to hold true for determinants, fail to be true for permanents. For our purposes it is enough to note that it is multi-linear and symmetric. For more details concerning permanents and their properties we refer the reader to [47].

We will prove the following result:

Proposition 3.1.7. *Let $\Omega \Subset \Sigma$ be as in Hypothesis 1.3.1. Let $\delta > 0$ be as in Hypothesis 1.3.2 and let $\Gamma = C - B^* A^{-1} B$. Moreover, let $D(\Omega, C_2)$ be as in (1.3.4). Then, there exists a smooth function*

$$D^{\delta}(z,w;h) = \frac{\operatorname{perm}\Gamma(z,w;h) \ + \mathcal{O}\left(\mathrm{e}^{-\frac{1}{Ch}} + \delta h^{-\frac{51}{10}}\right)}{\pi^2\left(\sqrt{\det A(z,w;h)} + \mathcal{O}\left(\delta h^{-\frac{3}{2}}\right)\right)^2} + \mathcal{O}\left(\mathrm{e}^{-\frac{D}{h^2}}\right).$$

and there exists a constant $C_2 > 0$ such that for all $\varphi \in \mathscr{C}_0(\Omega^2 \backslash D_h(\Omega, C_2))$

$$\int_{\mathbb{C}^2} \varphi(z,w) d\nu(z,w) = \int_{\mathbb{C}^2} \varphi(z,w) D(z,w,h,\delta) L(d(z,w)).$$

Remark 3.1.8. The proof of Proposition 3.1.7 will take up most of the rest of this chapter. Therefore we give a short overview on how we will proceed:

In Section 3.2, we give a formula for the scalar product $(X(z)|X(w))$ by constructing holomorphic quasimodes for the operators $(P_h - z)$ and $(P_h - z)^*$ to approximate the eigenfunction e_0 and f_0, and by using the method of stationary phase.

In Section 3.3, we will use this formula to study the invertibility of the matrices G, A and Γ. Furthermore, we will study the permanent of Γ.

In Section 3.4, we give a proof of Proposition 3.1.7.

3.2 | Stationary Phase

In this section we are interested in the scalar product $(X(z)|X(w))$. Recall from Definition 3.1.2 that the vector $X(z)$, $z \in \Omega$, is given by $X_{j,k} = \widehat{e}_0(z;k)\overline{\widehat{f}_0(z;j)}$, where e_0 and f_0 are the eigenfunctions of the operators $Q(z)$ and $\widetilde{Q}(z)$, respectively, associated to their first eigenvalue t_0^2.

The Fourier coefficients $\widehat{e}_0(z;k), \widehat{f}_0(z;j)$ and their z- and $\overline{z}$-derivatives are of order $\mathcal{O}(|k|^{-\infty})$, $\mathcal{O}(|j|^{-\infty})$, for $|j|,|k| \geq C/h$ with $C > 0$ large enough (cf Proposition 2.3.3 and 2.3.4). The Parseval identity implies that for $z, w \in \Omega$

$$(X(z)|X(w)) = (e_0(z)|e_0(w))(f_0(w)|f_0(z)) + \mathcal{O}_{\mathscr{C}^\infty}(h^\infty). \tag{3.2.1}$$

The aim of this section is to prove the following result:

Proposition 3.2.1. *Let $\Omega \Subset \Sigma$ be as in Hypothesis 1.3.1 and let $x_\pm(z)$ be as in (1.1.14). Furthermore, for $z \in \Omega$ let $\sigma(z)$ denote the Lebesgue density of the direct image of the symplectic volume form on T^*S^1 under the principal symbol p, i.e. $\sigma(z)L(dz) = p_*(d\xi\wedge dx)$.*

Then, there exists a constant $C > 0$ such that for all $(z,w) \in \Delta_\Omega(C) := \{(z,w)\in\Omega^2;\ |z-w| < 1/C\}$

$$(X(z)|X(w)) = \mathrm{e}^{-\frac{1}{h}\Phi(z;h) - \frac{1}{h}\Phi(w;h)} \mathrm{e}^{\frac{2}{h}\Psi(z,w;h)} + \mathcal{O}_{\mathscr{C}^\infty}\left(h^\infty\right)$$

where:

- *$\Phi(\cdot;h):\Omega\to\mathbb{R}$ is a family of smooth functions depending only on $i\operatorname{Im}z$, which satisfy*
$$\begin{aligned}\Phi(z;h)=&\operatorname{Im}\int_{x_+(z)}^{x_0}(z-g(y))dy-\operatorname{Im}\int_{x_-(z)}^{y_0}(z-g(y))dy\\&+\frac{h}{4}\left[\ln\left(\frac{\pi h}{-\operatorname{Im}g'(x_+(z))}\right)+\ln\left(\frac{\pi h}{\operatorname{Im}g'(x_-(z))}\right)\right]+\mathcal{O}(h^2).\end{aligned}$$
and
$$\partial^2_{z\overline{z}}\Phi(z;h)=\frac{1}{4}\sigma(z)+\mathcal{O}(h).$$
- *$\Psi(\cdot,\cdot;h):\Delta_\Omega(C)\to\mathbb{C}$ is a family of smooth functions which are almost z-holomorphic and almost w-anti-holomorphic extensions from the diagonal $\Delta:=\{(z,z);z\in\Omega\}\subset\Delta_\Omega(C)$ of $\Phi(z;h)$, i.e.*
$$\Psi(z,z;h)=\Phi\left(\frac{1}{2}(z-\overline{z});h\right),\ \partial_{\overline{z}}\Psi,\partial_w\Psi=\mathcal{O}(|z-w|^\infty).$$
Moreover, we have that $\Psi(z,z)=\Phi(z)$ and for $z,w\in\Delta_\Omega(C)$ with $|z-w|\ll1$,
$$\begin{aligned}\Psi(z,w;h)=&\sum_{|\alpha+\beta|\le2}\frac{1}{2^{|\alpha+\beta|}\alpha!\beta!}\partial_z^\alpha\partial_{\overline{z}}^\beta\Phi\left(\frac{z+w}{2};h\right)(z-w)^\alpha(\overline{w-z})^\beta\\&+\mathcal{O}(|z-w|^3+h^\infty),\end{aligned}$$
and
$$\begin{aligned}&2\operatorname{Re}\Psi(z,w;h)-\Phi(z;h)-\Phi(w;h)\\&=-\partial^2_{z\overline{z}}\Phi\left(\frac{z+w}{2};h\right)|z-w|^2(1+\mathcal{O}(|z-w|+h^\infty));\end{aligned}$$
- *the function $\Psi(z,w;h)$ has the following symmetries:*
$$\Psi(z,w;h)=\overline{\Psi(w,z;h)}\quad\text{and}\quad(\partial_z\Psi)(z,w;h)=\overline{(\partial_{\overline{w}}\Psi)(w,z;h)}.$$

Let us give some remarks on the above results: Note that the formula for Ψ stated above is simply a special case of the more general Taylor expansion
$$\begin{aligned}\Psi(z_0+\zeta,z_0+\omega;h)=&\sum_{|\alpha+\beta|\le2}\frac{1}{2^{|\alpha+\beta|}\alpha!\beta!}\partial_z^\alpha\partial_{\overline{z}}^\beta\Phi(z_0;h)\zeta^\alpha\overline{\omega}^\beta\\&+\mathcal{O}((\zeta,\omega)^3+h^\infty),\end{aligned}$$
with $z_0\in\Omega$ and $|\zeta|,|\omega|\ll1$.

Remark 3.2.2. Note that the formula for $(X(z)|X(w))$ is quite close to the notion of a Bergman kernel (see for example [87, Sec. 13.3]). However, we will not use this notion in the sequel.

Next, we define for $(z,w)\in\Delta_\Omega(C)$, as in Proposition 3.2.1,
$$\begin{aligned}-K(z,w):&=2\operatorname{Re}\Psi(z,w;h)-\Phi(z;h)-\Phi(w;h)\\&=-\left(\sigma\left(\frac{z+w}{2}\right)+\mathcal{O}(h)\right)\frac{|z-w|^2}{4}(1+\mathcal{O}(|z-w|+h^\infty)).\end{aligned}\tag{3.2.2}$$
From the above Proposition we can immediately deduce some growth properties of certain quantities that will be become important in the sequel.

Corollary 3.2.3. *Under the assumptions of Proposition 3.2.1, we have that*

- $|(X(z)|X(w))|=e^{-\frac{K(z,w)}{h}}+\mathcal{O}_{\mathcal{C}^\infty}(h^\infty)$;
- $$\begin{aligned}&\|X(z)\|^2\|X(w)\|^2\pm|(X(z)|X(w))|^2\\&=\left(1\pm e^{-\frac{2K(z,w)}{h}}\right)+\mathcal{O}_{\mathcal{C}^\infty}\left(h^\infty\right);\end{aligned}$$
- $$\begin{aligned}&\|X(z)\|^2\|X(w)\|^2|(X(z)|X(w))|^2\\&=e^{-\frac{2K(z,w)}{h}}+\mathcal{O}_{\mathcal{C}^\infty}\left(h^\infty\right).\end{aligned}$$

To prove Proposition 3.2.1, we will study the scalar products $(e_0(z)|e_0(w))$ and $(f_0(w)|f_0(z))$.

3.2.1 – The Scalar Product $(e_0(z)|e_0(w))$

We will prove

Proposition 3.2.4. *Let $\Omega \Subset \Sigma$ be as in Hypothesis 1.3.1 and let $x_+(z)$ be as in (1.1.14). Then, there exists a constant $C>0$ such that for all $(z,w) \in \Delta_\Omega(C) := \{(z,w) \in \Omega^2;\ |z-w| < 1/C\}$*

$$(e_0(z)|e_0(w)) = \mathrm{e}^{-\frac{1}{h}\Phi_1(z;h)}\mathrm{e}^{-\frac{1}{h}\Phi_1(w;h)}\mathrm{e}^{\frac{2}{h}\Psi_1(z,w;h)} + \mathscr{O}\left(h^\infty\right), \tag{3.2.3}$$

where:

- *$\Phi_1(\cdot;h) : \Omega \to \mathbb{R}$ is a family of smooth functions depending only on $i\mathrm{Im}\, z$, which satisfy*
$$\Phi_1(z;h) = \mathrm{Im}\int_{x_+(\mathrm{Im}\, z)}^{x_0} (z-g(y))dy + \frac{h}{4}\ln\left(\frac{\pi h}{-\mathrm{Im}\, g'(x_+)}\right) + \mathscr{O}(h^2).$$

- *$\Psi_1(\cdot,\cdot;h) : \Delta_\Omega(C) \to \mathbb{C}$ is a family of smooth functions which are almost z-holomorphic and almost w-anti-holomorphic extensions from the diagonal $\Delta := \{(z,z); z \in \Omega\} \subset \Delta_\Omega(C)$ of $\Phi_1(z;h)$, i.e.*
$$\Psi_1(z,z;h) = \Phi_1\left(\frac{1}{2}(z-\overline{z});h\right),\ \partial_{\overline{z}}\Psi_1, \partial_w\Psi_1 = \mathscr{O}(|z-w|^\infty).$$
Moreover, for $z,w \in \Delta_\Omega(C)$ with $|z-w| \ll 1$, one has that
$$\begin{aligned}\Psi_1(z,w;h) = &\sum_{|\alpha+\beta|\leq 2}\frac{1}{2^{|\alpha+\beta|}\alpha!\beta!}\partial_z^\alpha\partial_{\overline{z}}^\beta\Phi_1\left(\frac{z+w}{2};h\right)(z-w)^\alpha\,\overline{(w-z)}^\beta \\ &+\mathscr{O}(|z-w|^3 + h^\infty),\end{aligned}$$
and that
$$\begin{aligned}&2\mathrm{Re}\,\Psi_1(z,w;h) - \Phi_1(z;h) - \Phi_1(w;h) \\ &\quad = -\partial_z\partial_{\overline{z}}\Phi_1\left(\frac{z+w}{2};h\right)|z-w|^2(1+\mathscr{O}(|z-w|+h^\infty));\end{aligned}$$

- *the function $\Psi_1(z,w;h)$ has the following symmetries:*
$$\Psi_1(z,w;h) = \overline{\Psi_1(w,z;h)} \quad \text{and} \quad (\partial_z\Psi_1)(z,w;h) = \overline{(\partial_{\overline{w}}\Psi_1)(w,z;h)}.$$

To prove Proposition 3.2.4, we begin by constructing an oscillating function to approximate $e_0(z)$. Let us recall from Section 1.1.1 that the points $a,b \in S^1$ denote the minimum and the maximum of $\mathrm{Im}\, g(x)$ and that for $z \in \Omega$ the points $x_\pm(z) \in S^1$ are the unique solutions to the equation $\mathrm{Im}\, g(x) = \mathrm{Im}\, z$. Furthermore, we will identify frequently S^1 with the interval $[b-2\pi, b[$. Moreover, let us recall that by the natural projection $\Pi : \mathbb{R} \to S^1 = \mathbb{R}/2\pi\mathbb{Z}$ we identify the points $x_\pm, a, b \in S^1$ with points $x_\pm, a, b \in \mathbb{R}$ such that $b-2\pi < x_+ < a < x_- < b$.

Let $K_+ \subset]b-2\pi, a[$ be an open interval such that $x_+(z) \in K_+$ for all $z \in \Omega$. Let $\chi \in \mathscr{C}_0^\infty(]b-2\pi, a[)$ and define for $x \in \mathbb{R}$
$$\widetilde{e}_0(x,z) := \chi(x)\exp\left(\frac{i}{h}\psi_+(x,z)\right). \tag{3.2.4}$$
where, for a fixed $x_0 \in K_+$,
$$\psi_+(x,z) := \int_{x_0}^x \left(z-g(y)\right)dy. \tag{3.2.5}$$

Remark 3.2.5. Note that the function $u = \exp(i\psi_+(x,z)/h)$ is solution to $(P_h - z)u = 0$ on $\mathrm{supp}\,\chi$, since the phase function ψ_+ satisfies the eikonal equation

$$p(x, \partial_x\psi_+) = z.$$

Furthermore, let us remark that $\widetilde{e}_0(x,z)$ depends holomorphically on z.

Next, we are interested in the L^2-norm of $\widetilde{e}_0$.

Lemma 3.2.6. *Let $\Omega \Subset \Sigma$ be as in Hypothesis 1.3.1. Then, there exists a family of smooth functions $\Phi_1(\cdot;h):\Omega \to \mathbb{R}$, such that*

$$\Phi_1(z;h) = \Phi_1(i\operatorname{Im} z;h) = \operatorname{Im}\int_{x_+(\operatorname{Im} z)}^{x_0}(z-g(y))dy + \frac{h}{4}\ln\left(\frac{\pi h}{-\operatorname{Im} g'(x_+)}\right) + \mathcal{O}(h^2)$$

and

$$\|\widetilde{e}_0(z)\|^2 = \exp\left\{\frac{2}{h}\Phi_1(z;h)\right\}.$$

Proof. In view of the definition of $\widetilde{e}_0(z)$, see (3.2.4) and (3.2.5), one gets that

$$\|\widetilde{e}_0(z)\|^2 = \int \chi(x)\mathrm{e}^{\frac{i}{h}(\psi_+(x,z)-\overline{\psi}_+(x,z))}dx = \int \chi(x)\mathrm{e}^{-\frac{2}{h}\operatorname{Im}\psi_+(x,z)}dx.$$

The critical point for $\operatorname{Im}\psi_+(x,z)$ is given by the equation

$$\operatorname{Im}\partial_x\psi_+(x,z) = \operatorname{Im} z - \operatorname{Im} g(x) = 0, \quad x \in \operatorname{supp}\chi.$$

The critical point, given by $x_+(\operatorname{Im} z)$, is unique and it satisfies $\operatorname{Im} g'(x_+(\operatorname{Im} z)) < 0$, see (1.1.14). This implies in particular that the critical point is non-degenerate. More precisely,

$$\operatorname{Im}(\partial^2_{xx}\psi_+)(x_+,z) = -\operatorname{Im} g'(x_+) > 0. \tag{3.2.6}$$

The critical value of $\operatorname{Im}\psi_+$ is given by

$$\operatorname{Im}\psi_+(x_+(\operatorname{Im} z),z) = \operatorname{Im}\int_{x_0}^{x_+(\operatorname{Im} z)}(z-g(y))dy \le 0.$$

Using the method of stationary phase, one gets

$$\begin{aligned}\|\widetilde{e}_0(z)\|^2 &= \sqrt{\frac{\pi h}{\operatorname{Im}(\partial^2_{xx}\psi_+)(x_+,z)}}(1+\mathcal{O}(h))\exp\left\{-\frac{2\operatorname{Im}\psi_+(x_+,z)}{h}\right\}\\ &=: \exp\left\{\frac{2}{h}\Phi_1(z;h)\right\},\end{aligned}$$

where Φ_1 is smooth in z. Using (3.2.6), one gets that

$$\Phi_1(z;h) = \operatorname{Im}\int_{x_+(\operatorname{Im} z)}^{x_0}(z-g(y))dy + \frac{h}{4}\ln\left(\frac{\pi h}{-\operatorname{Im} g'(x_+)}\right) + \mathcal{O}(h^2). \qquad \square$$

Recall from (1.2.7) that the function e_0 is an eigenfunction of the operator $Q(z)$ (cf Section 1.2.2) corresponding to its first eigenvalue t_0^2. We set

$$e_0(z) = \frac{\Pi_{t_0^2}\left(\mathrm{e}^{-\frac{1}{h}\Phi_1(z;h)}\widetilde{e}_0(z)\right)}{\left\|\Pi_{t_0^2}\left(\mathrm{e}^{-\frac{1}{h}\Phi_1(z;h)}\widetilde{e}_0(z)\right)\right\|},$$

where $\Pi_{t_0^2}: L^2(S^1) \to \mathbb{C}e_0$ denotes the spectral projection for $Q(z)$ onto the eigenspace associated with t_0^2.

Next, we prove that up to an exponentially small error in $1/h$, e_0 is given by the normalization of $\widetilde{e}_0$.

Lemma 3.2.7. *Let $\Omega \Subset \Sigma$ be as in Hypothesis 1.3.1. Then, there exists a constant $C > 0$ such that for all $z \in \Omega$ and all $\alpha \in \mathbb{N}^2$*

$$\left\|\partial^\alpha_{z,\bar{z}}\left(e_0(z) - \mathrm{e}^{-\frac{1}{h}\Phi_1(z;h)}\widetilde{e}_0(z)\right)\right\| = \mathcal{O}\left(h^{-|\alpha|}\mathrm{e}^{-\frac{1}{Ch}}\right).$$

Proof. The proof of the lemma is similar to the proof of Proposition 2.1.11. □

This result implies that

$$(e_0(z)|e_0(w)) = \mathrm{e}^{-\frac{1}{h}\Phi_1(z;h)-\frac{1}{h}\Phi_1(w;h)}(\widetilde{e}_0(z)|\widetilde{e}_0(w)) + \mathcal{O}_{\mathscr{C}^\infty}\left(\mathrm{e}^{-\frac{1}{Ch}}\right). \tag{3.2.7}$$

By Remark 3.2.5, $(\widetilde{e}_0(z)|\widetilde{e}_0(w))$ is holomorphic in z and anti-holomorphic in w. We can study this scalar product by the method of stationary phase:

Proof of Proposition 3.2.4. In view of (3.2.7), it remains to study the oscillatory integral

$$I(z,w) := (\widetilde{e}_0(z)|\widetilde{e}_0(w)) = \int \chi(x)\exp\left(\frac{i}{h}\Psi_+(x,z,w)\right)dx, \tag{3.2.8}$$

where $\widetilde{e}_0(x,z)$ is given in (3.2.4) and Ψ_+ is defined by

$$\Psi_+(x,z,w) := \psi_+(x,z) - \overline{\psi_+(x,w)},\ z,w \in \Omega. \tag{3.2.9}$$

Using (3.2.5),

$$\Psi_+(x,z,w) = \int_{x_0}^{x}\mathrm{Re}\,(z-w)\,dy + 2i\int_{x_0}^{x}\left[\mathrm{Im}\left(\frac{z+w}{2}\right) - \mathrm{Im}\,g(y)\right]dy. \tag{3.2.10}$$

Since the imaginary part of Ψ_+ can be negative, we shift the phase function by the minimum of $\mathrm{Im}\,\Psi_+$.

Minimum of $\mathrm{Im}\,\Psi_+$**.** The critical points of the function $x \mapsto \mathrm{Im}\,\Psi(x,z,w)$ are given by the equation $\mathrm{Im}\,(\frac{z+w}{2}) = \mathrm{Im}\,g(x)$. Since Ω is convex, this equation has, for $|z-w|$ small enough, on the support of χ the unique solution $x_+(\frac{z+w}{2}) \in \mathbb{R}$ and it satisfies $\mathrm{Im}\,g'(x_+(\frac{z+w}{2})) < 0$ (cf. (1.1.14)). Moreover, it depends smoothly on z and w since g is smooth. Therefore,

$$(\partial_{xx}^2\mathrm{Im}\,\Psi_+)\left(x_+\left(\frac{z+w}{2}\right),z,w\right) = -2\mathrm{Im}\,g_x'\left(x_+\left(\frac{z+w}{2}\right)\right) > 0,$$

which implies that $x_+(\frac{z+w}{2})$ is a minimum point, and that

$$\begin{aligned} 2\lambda := 2\lambda(z,w) :&= \mathrm{Im}\,\Psi_+\left(x_+\left(\frac{z+w}{2}\right),z,w\right) \\ &= 2\int_{x_0}^{x_+(\frac{z+w}{2})}\left[\mathrm{Im}\left(\frac{z+w}{2}\right) - \mathrm{Im}\,g(y)\right]dy \le 0. \end{aligned} \tag{3.2.11}$$

We define $\Theta_+(x,z,w) := \Psi_+(x,z,w) - i\lambda$, and notice that $\mathrm{Im}\,\Theta_+(x,z,w) \ge 0$. Hence, we can write (3.2.8) as follows:

$$I(z,w) = \mathrm{e}^{-\frac{2\lambda}{h}}\int \chi(x)\exp\left(\frac{i}{h}\Theta_+(x,z,w)\right)dx. \tag{3.2.12}$$

To study $I(z,w)$ by the method of stationary phase, we are interested in the critical points of Θ_+.

Critical points of Θ_+**.** Clearly they are the same as for $\Psi_+(x,z,w)$. Note that for $z=w$ one has that

$$\Psi_+(x,z,z) = 2i\mathrm{Im}\int_{x_0}^{x}(z-g(y))dy$$

which has, on the support of χ, the unique critical point x_+ and it satisfies $\mathrm{Im}\,g'(x_+) < 0$ (cf. (1.1.14)). Therefore,

$$\mathrm{Im}\,(\partial_{xx}^2\Psi_+)(x_+(z),z,z) = -2\mathrm{Im}\,g_x'(x_+(z)) > 0$$

which implies that x_+ is a non-degenerate critical point.

In the case where $z \neq w$ the situation is more complicated. By (3.2.10) we see that if $\mathrm{Re}\,(z-w) = 0$, for $|z-w|$ small enough, the critical point is real and given by $x_+(\frac{z+w}{2})$, i.e. the minimum point of $\mathrm{Im}\,\Psi_+$.

However, if $\mathrm{Re}\,(z-w) \neq 0$, we need to consider an almost x-analytic extension of Ψ_+, which we shall denote by $\widetilde{\Psi}_+$. As described in [48], the "critical point" of $\widetilde{\Psi}_+$ is then given by

$$\partial_x \widetilde{\Psi}_+(x,z,w) = 0,$$

and we will see, by the following result, that it "moves" to the complex plane.

Lemma 3.2.8. *Let $\Omega \Subset \Sigma$ be as in (1.3.1). Let χ be as in (3.2.4) and let p be the principal symbol of P_h (cf (1.1.7)). Let $x_+(z)$ be as in (1.1.14). Furthermore, let $\widetilde{\psi}_+$ denote an almost analytic extension of ψ_+ to a small complex neighborhood of the support of χ, and define $\widetilde{\psi}_+^*(x) := \overline{\widetilde{\psi}_+(\overline{x})}$. Then, the there exists a $C>0$ such that for $(z,w) \in \Delta_\Omega(C)$ the function*

$$\partial_x \widetilde{\Psi}_+(x,z,w) = \partial_x \widetilde{\psi}_+(x,z) - (\partial_x \widetilde{\psi}_+)^*(x,w)$$

has exactly one zero, $x_+^c(z,w)$, and:

- *it depends almost holomorphically on z and almost anti-holomorphically w at the diagonal Δ, i.e.*

$$\partial_w x_+^c(z,w), \partial_{\overline{z}} x_+^c(z,w) = \mathcal{O}(|z-w|^\infty);$$

- *it is non-degenerate in the sense that*

$$(\partial_{xx}^2 \widetilde{\Psi}_+)(x_+^c(z,w),z,w) \neq 0;$$

- *for $z,w \in \Omega$ with $|z-w| < 1/C$, $C>1$ large enough, one has*

$$x_+^c(z,w) = x_+\left(\frac{z+w}{2}\right) - \frac{\mathrm{Re}\,(z-w)}{\{p,\overline{p}\}(\rho_+\left(\frac{z+w}{2}\right))} + \mathcal{O}(|z-w|^2).$$

Remark 3.2.9. The proof of Lemma 3.2.8 will be given after the proof of Proposition 3.2.4.

Let $\widetilde{\Psi}_+$ denote an almost x-analytic extension of Ψ_+. Using the method of stationary phase for complex-valued phase functions (cf. Theorem 2.3 in [48, p.148]) and Lemma 3.2.8, one gets that

$$I(z,w) = \exp\left\{\frac{2\Psi_1(z,w;h)}{h}\right\} + \mathcal{O}\left(h^\infty\right) \mathrm{e}^{-\frac{2\lambda}{h}}. \tag{3.2.13}$$

Using that Lemma 3.2.6 and (3.2.11) imply $\lambda(z,w) + \Phi(z;h) + \Phi(w;h) \geq 0$, we obtain (3.2.3) from the above and (3.2.7).

In (3.2.13), $2\Psi_1(z,w)$ is given by the critical value of $i\widetilde{\Psi}_+$ and by the logarithm of the amplitude $c(z,w,h)$, given by the stationary phase method, i.e.

$$2\Psi_1(z,w;h) = i\widetilde{\Psi}_+(x_+^c(z,w),z,w) + h \ln c(z,w,h)$$

and $c(z,w,h) \sim c_0(z,w) + hc_1(z,w) + \dots$ which depends smoothly on z and w in the sense that all z-,$\bar{z}$-,w- and $\bar{w}$-derivatives remain bounded as $h \to 0$. $\widetilde{\Psi}_+(x,z,w)$ is by definition z-holomorphic, w-anti-holomorphic and smooth in x. By Lemma 3.2.8, we know that the critical point $x_+^c(z,w)$ is almost z-holomorphic and almost w-anti-holomorphic in $\Delta_\Omega(C)$, a small neighborhood of the diagonal $z=w$. Hence, Ψ is almost z-holomorphic and almost w-anti-holomorphic in $\Delta_\Omega(C)$.

Equivalently, Ψ is an almost z-holomorphic and almost w-anti-holomorphic extension from the diagonal of $\Psi_1(z,z;h)$. Since $\Psi_1(z,z;h) = \Phi_1(z;h)$, we obtain by Taylor expansion up to order 2 of Ψ at $(\frac{z+w}{2}, \frac{z+w}{2})$, that

$$\begin{aligned}\Psi_1(z,w;h) = &\sum_{|\alpha+\beta|\leq 2} \frac{1}{2^{|\alpha+\beta|}\alpha!\beta!} \partial_z^\alpha \partial_{\overline{z}}^\beta \Phi_1\left(\frac{z+w}{2};h\right)(z-w)^\alpha \left(\overline{w-z}\right)^\beta \\ &+ \mathcal{O}(|z-w|^3 + h^\infty),\end{aligned}$$

for $|z-w|$ small enough. Similarly,

$$\Phi_1(z;h) = \sum_{|\alpha+\beta|\leq 2} \frac{1}{2^{|\alpha+\beta|}\alpha!\beta!} \partial_z^\alpha \partial_{\overline{z}}^\beta \Phi_1\left(\frac{z+w}{2};h\right)(z-w)^\alpha (\overline{z-w})^\beta$$
$$+\mathcal{O}(|z-w|^3+h^\infty),$$

which implies that

$$2\mathrm{Re}\,\Psi_1(z,w;h) = \Phi_1(z;h)+\Phi_1(w;h) - \partial_z^\alpha \partial_{\overline{z}}^\beta \Phi_1\left(\frac{z+w}{2};h\right)|z-w|^2$$
$$+\mathcal{O}(|z-w|^3+h^\infty),$$

concluding the proof of the second point of the proposition.

Finally, let us give a proof of the stated symmetries. The fact that $\Psi_1(z,w;h) = \overline{\Psi_1(w,z;h)}$ follows directly from the fact that $(e_0(z)|e_0(w)) = \overline{(e_0(w)|e_0(z))}$. One then computes that

$$(\partial_z\Psi_1)(z,w;h) = \partial_z\Psi_1(z,w;h) = \overline{\partial_{\overline{z}}\Psi_1(w,z;h)} = \overline{(\partial_{\overline{w}}\Psi_1)(w,z;h)}$$

which concludes the proof of the Proposition. □

Proof of Lemma 3.2.8. We are interested in the solutions of the following equation:

$$0 = (\partial_x\widetilde{\psi}_+)(x,z) - (\partial_x\widetilde{\psi}_+)^*(x,w) = z-\overline{w}-\widetilde{g}(x)+\widetilde{g}^*(x), \tag{3.2.14}$$

where $\widetilde{g}$ denotes an almost analytic extension of g. Since $\mathrm{dist}(\Omega,\partial\Sigma) > 1/C$, it follows from the assumptions on g that $\mathrm{Im}\,g'(x) > 0$ for all $x \in \overline{x_+(\Omega)} \subset \mathbb{R}$. Since g depends smoothly on x, there exists a small complex open neighborhood $V \subset \mathbb{C}$ of $\overline{x_+(\Omega)}$ such that $\overline{x_+(\Omega)} \subset (V\cap\mathbb{R})$ and such that for all $x \in V$

$$\widetilde{g}_x'(x) - \overline{\widetilde{g}_x'(\overline{x})} \neq 0, \quad \widetilde{g}_{\overline{x}}'(x) - \overline{\widetilde{g}_{\overline{x}}'(\overline{x})} = \mathcal{O}(|\mathrm{Im}\,x|^\infty).$$

Thus, it follows by the implicit function theorem, that for $(z,w) \in \Delta_\Omega(C)$, with $C>0$ large enough, there exists a unique solution $x_+^c(z,w)$ to (3.2.14) and it depends smoothly on $(z,w) \in \Delta_\Omega(C)$. Furthermore, we have that $x_+^c(z,z) = x_+(z) \in \mathbb{R}$. Taking the z- and $\overline{z}$- derivative of (3.2.14) at the critical point x_+^c yields that

$$\partial_z x_+^c(z,w) = \frac{1+\mathcal{O}(|\mathrm{Im}\,x_+^c(z,w)|^\infty)}{(\partial_x\widetilde{g})(x_+^c(z,w)) - (\partial_x\widetilde{g})^*(x_+^c(z,w))},$$
$$\partial_{\overline{z}} x_+^c(z,w) = \frac{\mathcal{O}(|\mathrm{Im}\,x_+^c(z,w)|^\infty)}{(\partial_x\widetilde{g})(x_+^c(z,w)) - (\partial_x\widetilde{g})^*(x_+^c(z,w))} \tag{3.2.15}$$

and similarly that

$$\partial_{\overline{w}} x_+^c(z,w) = \frac{-1+\mathcal{O}(|\mathrm{Im}\,x_+^c(z,w)|^\infty)}{(\partial_x\widetilde{g})(x_+^c(z,w)) - (\partial_x\widetilde{g})^*(x_+^c(z,w))},$$
$$\partial_w x_+^c(z,w) = \frac{\mathcal{O}(|\mathrm{Im}\,x_+^c(z,w)|^\infty)}{(\partial_x\widetilde{g})(x_+^c(z,w)) - (\partial_x\widetilde{g})^*(x_+^c(z,w))}. \tag{3.2.16}$$

Using that $\mathrm{Im}\,x_+^c(z,z) = 0$, one calculates that for $z=w$ we have that

$$(\partial_z x_+^c)(z,z) = \partial_z x_+(z) = -(\partial_{\overline{w}} x_+^c)(z,z),$$
$$\text{and } (\partial_{\overline{z}} x_+^c)(z,z) = 0 = (\partial_w x_+^c)(z,z), \tag{3.2.17}$$

where

$$\partial_z x_+(z) = \frac{1}{2i\mathrm{Im}\,g'(x_+(z))}.$$

Taylor's theorem implies that

$$x_+^c(z+\zeta, z+\omega) = x_+(z) + \frac{\zeta - \overline{\omega}}{2i\mathrm{Im}\, g'(x_+(z))} + \mathcal{O}((\zeta,\omega)^2).$$

Recall that the principal symbol of the operator P_h is given by $p(\rho) = \xi + g(x)$ (cf (1.1.7)), which implies that $\{p, \overline{p}\}(\rho_\pm(z)) = -2i\mathrm{Im}\, g'(x_\pm(z))$. To conclude the symmetric form of the Taylor expansion stated in the Lemma, we expand around the point $(\frac{z+w}{2}, \frac{z+w}{2})$, for $|z-w|$ small enough, with $\zeta = \frac{z-w}{2}$ and $\omega = -\frac{z-w}{2}$, which is possible since Ω is by (1.3.1) assumed to be convex.

Finally, by taking the imaginary part of the Taylor expansion of x_+^c, we conclude by (3.2.15) and (3.2.16) that

$$\partial_w x_+^c(z,w), \partial_{\overline{z}} x_+^c(z,w) = \mathcal{O}(|z-w|^\infty).$$ □

3.2.2 – The Scalar Product $(f_0(w)|f_0(z))$

We have, as in Section 3.2.1,

Proposition 3.2.10. *Let $\Omega \Subset \Sigma$ be as in Hypothesis 1.3.1 and let $x_-(z)$ be as in (1.1.14). Then, there exists a constant $C>0$ such that for all $(z,w) \in \Delta_\Omega(C) := \{(z,w) \in \Omega^2;\ |z-w| < 1/C\}$*

$$(f_0(w)|f_0(z)) = \mathrm{e}^{-\frac{1}{h}\Phi_2(z;h)}\mathrm{e}^{-\frac{1}{h}\Phi_2(w;h)}\mathrm{e}^{\frac{2}{h}\Psi_2(z,w;h)} + \mathcal{O}\left(h^\infty\right),$$

where:

- $\Phi_2(\cdot;h): \Omega \to \mathbb{R}$ *is a family of smooth functions depending only on* $\mathrm{Im}\, z$*, which satisfy*

$$\Phi_2(z;h) = -\mathrm{Im}\int_{x_-(z)}^{x_0} (z - g(y))dy + \frac{h}{4}\ln\left(\frac{\pi h}{\mathrm{Im}\, g'(x_-(z))}\right) + \mathcal{O}(h^2).$$

- $\Psi_2(\cdot,\cdot;h): \Delta_\Omega(C) \to \mathbb{C}$ *is a family of smooth functions which are almost z-holomorphic and almost w-anti-holomorphic extensions from the diagonal $\Delta := \{(z,z); z \in \Omega\} \subset \Delta_\Omega(C)$ of $\Phi_2(z;h)$, i.e.*

$$\partial_{\overline{z}}\Psi_2, \partial_w\Psi_2 = \mathcal{O}(|z-w|^\infty),\ \Psi_2(z,z;h) = \Phi_2\left(\frac{1}{2}(z-\overline{z});h\right)$$

Moreover, for $z,w \in \Delta_\Omega(C)$ with $|z-w| \ll 1$, one has that

$$\Psi_2(z,w;h) = \sum_{|\alpha+\beta|\le 2} \frac{1}{2^{|\alpha+\beta|}\alpha!\beta!}\partial_z^\alpha\partial_{\overline{z}}^\beta\Phi_2\left(\frac{z+w}{2};h\right)(z-w)^\alpha\overline{(w-z)}^\beta + \mathcal{O}(|z-w|^3 + h^\infty),$$

and that

$$2\mathrm{Re}\,\Psi_2(z,w;h) - \Phi_2(z;h) - \Phi_2(w;h) = -\partial_z\partial_{\overline{z}}\Phi_2\left(\frac{z+w}{2};h\right)|z-w|^2(1+\mathcal{O}(|z-w| + h^\infty));$$

- *the function $\Psi_2(z,w;h)$ has the following symmetries:*

$$\Psi_2(z,w;h) = \overline{\Psi_2(w,z;h)} \quad \text{and} \quad (\partial_z\Psi_2)(z,w;h) = \overline{(\partial_{\overline{w}}\Psi_2)(w,z;h)}.$$

3.2.3 – Link with the symplectic volume

Before the proof of Proposition 3.2.1, let us give a short description of the connection between the functions $\Phi_1(z;h)$, $\Phi_2(z;h)$ in Proposition 3.2.4, 3.2.10, and the symplectic volume form on the phase space T^*S^1.

Proposition 3.2.11. *Let $z \in \Omega \Subset \Sigma$ be as in (1.3.1) and let Φ_1 and Φ_2 be as in Propositions 3.2.4 and 3.2.10. Furthermore, let p be the principal symbol of P_h (cf (1.1.7)), let $\rho_\pm \in T^*S^1$ be the two solutions to $p(\rho) = z$, see (1.1.14). Then,*

$$\begin{aligned}\sigma_h(z) :&= \left[(\partial^2_{z\overline{z}}\Psi_1)(z;h) + (\partial^2_{z\overline{z}}\Phi_2)(z;h)\right] \\ &= \frac{1}{4}\left(\frac{1}{\frac{1}{2i}\{\overline{p},p\}(\rho_-(z))} + \frac{1}{\frac{1}{2i}\{p,\overline{p}\}(\rho_+(z))}\right) + \mathcal{O}(h)\end{aligned}$$

is, up to an error of order h, one-fourth of the Lebesgue density of the direct image, under the principal symbol p, of the symplectic volume form $d\xi \wedge dx$ on T^*S^1, i.e.

$$\sigma_h(z)L(dz) = \frac{1}{4}p_*(d\xi\wedge dx) + \mathcal{O}(h)L(dz)$$

Proof. Using that $x_\pm(t)$, with $t = \operatorname{Im} z$, is the solution to the equation $\operatorname{Im} g(x_\pm(t)) = t$ with

$$\mp \operatorname{Im} g'_x(x_\pm(t)) < 0$$

(cf (1.1.14)), we get that

$$x'_\pm(t) = \pm\frac{1}{\operatorname{Im} g'_x(x_\pm(t))} < 0.$$

Using Propositions 3.2.4 and 3.2.10, one then computes that

$$(\partial^2_{z\overline{z}}\Phi_1)(z;h) + (\partial^2_{z\overline{z}}\Phi_2)(z;h) = \frac{1}{4}\left(\frac{1}{\operatorname{Im} g'_x(x_-(\operatorname{Im} z))} - \frac{1}{\operatorname{Im} g'_x(x_+(\operatorname{Im} z))}\right) + \mathcal{O}(h).$$

Since $-\frac{1}{2i}\{p,\overline{p}\}(\rho_\pm) = \operatorname{Im} g'_x(x_\pm)$, we conclude by Proposition 2.4.2 that

$$\left[\partial^2_{z\overline{z}}\Phi_1)(z;h) + (\partial^2_{z\overline{z}}\Phi_2)(z;h)\right]L(dz) = \frac{1}{4}p_*(d\xi\wedge dx) + \mathcal{O}(h)L(dz). \qquad \square$$

Proof of Proposition 3.2.1. The results follow immediately from (3.2.1) and the Propositions 3.2.4, 3.2.10 and 3.2.11. $\square$

3.3 | Gramian matrix

The aim of this section is to study the Gramian matrix G defined in (3.1.6) by

$$G := \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \in \mathbb{C}^{4\times 4},$$

where

$$A := \begin{pmatrix} (X(z)|X(z)) & (X(z)|X(w)) \\ (X(w)|X(z)) & (X(w)|X(w)) \end{pmatrix},$$

$$B := \begin{pmatrix} (X(z)|\partial_z X(z)) & (X(z)|\partial_w X(w)) \\ (X(w)|\partial_z X(z)) & (X(w)|\partial_w X(w)) \end{pmatrix},$$

$$C := \begin{pmatrix} (\partial_z X(z)|\partial_z X(z)) & (\partial_z X(z)|\partial_w X(w)) \\ (\partial_w X(w)|\partial_z X(z)) & (\partial_w X(w)|\partial_w X(w)) \end{pmatrix}.$$

The invertibility of the matrix G will be essential to the proof of Proposition 3.1.7. Indeed, we prove the following result.

Proposition 3.3.1. *Let $\Omega \Subset \Sigma$ be as in (1.3.1) and let $z, w \in \Omega$. Then,*

$$\det G(z,w) > 0 \quad \text{for} \quad h^{\frac{3}{5}} \ll |z-w| \ll 1.$$

To prove Proposition 3.3.1 we will first study the matrices A and, if A^{-1} exists, the matrix Γ given by the Shur complement formula applied to G, i.e.

$$\Gamma = C - B^* A^{-1} B. \tag{3.3.1}$$

3.3.1 – The matrix A

We begin by studying the determinant of A. It is non-zero if and only if the vectors $X(z)$ and $X(w)$ are not co-linear. In particular we are interested in a lower bound of this determinant for z and w close.

Proposition 3.3.2. *Let $\Omega \Subset \Sigma$ be as in Hypothesis 1.3.1. For $|z-w| \leq 1/C$, with $C > 1$ large enough (cf. Proposition 3.2.1), we have*

$$\det A(z,w) = 1 - \mathrm{e}^{-\frac{2K(z,w)}{h}} + \mathcal{O}_{\mathscr{C}^\infty}\left(h^\infty\right),$$

where $K(z,w)$ is as in (3.2.2). Moreover,

- *for $|z-w| \gg \sqrt{h \ln h^{-1}}$*

$$\det A(z,w) = 1 + \mathcal{O}\left(h^C\right),\ C \gg 1;$$

- *for $|z-w| \geq \frac{1}{\mathcal{O}(1)}\sqrt{h}$*

$$\det A \geq \frac{1}{\mathcal{O}(1)};$$

- *let $N > 1$ and let $C > 1$ be large enough, then for $\frac{1}{C}h^N \leq |z-w| \leq \frac{1}{C}\sqrt{h}$,*

$$\begin{aligned}\det A(z,w) &= \frac{|z-w|^2}{2h}\left(\sigma\left(\frac{z+w}{2}\right) + \mathcal{O}(h) + \mathcal{O}(|z-w|) + \mathcal{O}\left(\frac{|z-w|^2}{h}\right)\right) \\ &\quad + \mathcal{O}_{\mathscr{C}^\infty}\left(h^\infty\right) \\ &\geq \frac{h^{2N-1}}{\mathcal{O}(1)}.\end{aligned}$$

Since the matrix A is self-adjoint, we have a lower bound on the matrix norm of A by its smallest eigenvalue. Using Proposition 3.2.1 we see that $\operatorname{tr} A = 2 + \mathcal{O}(h^\infty)$ and one calculates that for a fixed $N > 1$ and for $|z-w| \geq \frac{h^N}{\mathcal{O}(1)}$ the two eigenvalues of A are given by

$$\lambda_{1,2}(z,w;h) = 1 \pm \mathrm{e}^{-\frac{K(z,w)}{h}} + \mathcal{O}(h^\infty).$$

By Taylor expansion we conclude the following result:

Corollary 3.3.3. *Under the assumptions of Proposition 3.3.2, we have that for $N \geq 1$ and $|z-w| \geq \frac{h^N}{\mathcal{O}(1)}$*

$$\min_{\lambda \in \sigma(A)} \lambda \geq \frac{h^{N-\frac{1}{2}}}{\mathcal{O}(1)}.$$

Proof of Proposition 3.3.2. By Corollary 3.2.3 and (3.2.2), one has that

$$\det A(z,w) = 1 - \mathrm{e}^{-\frac{2K(z,w)}{h}} + \mathcal{O}_{\mathscr{C}^\infty}\left(h^\infty\right),$$

with

$$K(z,w) = \left(\sigma\left(\frac{z+w}{2}\right) + \mathcal{O}(h)\right)\frac{|z-w|^2}{4}(1 + \mathcal{O}(|z-w| + h^\infty)).$$

The first two estimates are then an immediate consequence of the above formula. In the case where $|z-w| \leq \frac{1}{C}\sqrt{h}$, one computes, using Taylor's formula, that

$$\mathrm{e}^{-\frac{2K(z,w)}{h}} = 1 - \frac{|z-w|^2}{2h}\left(\sigma\left(\frac{z+w}{2}\right) + \mathcal{O}(h) + \mathcal{O}(|z-w|) + \mathcal{O}\left(\frac{|z-w|^2}{h}\right)\right),$$

which implies that

$$\begin{aligned}\det A(z,w) &= \frac{|z-w|^2}{2h}\left(\sigma\left(\frac{z+w}{2}\right)+\mathcal{O}(h)+\mathcal{O}(|z-w|)+\mathcal{O}\left(\frac{|z-w|^2}{h}\right)\right)\\ &\quad+\mathcal{O}_{\mathscr{C}^\infty}(h^\infty)\\ &\geq \frac{h^{2N-1}}{\mathcal{O}(1)}.\end{aligned}$$

□

3.3.2 – The matrix Γ

We prove the following result.

Proposition 3.3.4. *Let $\Omega \Subset \Sigma$ be as in (1.3.1), and let $D_\Omega(C)$ and $\Psi(z,w;h)$ for $(z,w)\in D_\Omega(C)$ be as in Proposition 3.2.1. Let Γ be as in (3.3.1). For $(z,w)\in D_\Omega(C)$ let $K(z,w)$ be as in (3.2.2) and define*

$$\begin{aligned}a_1 &:= a_1(z,w;h) := (\partial_z\Psi)(z,z;h)-(\partial_z\Psi)(z,w;h),\\ a_2 &:= a_2(z,w;h) := -a_1(w,z;h).\end{aligned}$$

Then, for $N>1$ and $\frac{1}{C}h^N\leq |z-w|$, with $C>1$ large enough, we have that

$$\begin{aligned}\Gamma =& \frac{-4}{h^2\left(1-\mathrm{e}^{-\frac{2}{h}K(z,w)}\right)}\begin{pmatrix} a_1\overline{a}_1\mathrm{e}^{-\frac{2}{h}K(z,w)} & a_1\overline{a}_2\mathrm{e}^{\frac{1}{h}(2i\operatorname{Im}\Psi(z,w)-K(z,w))}\\ a_2\overline{a}_1\mathrm{e}^{\frac{1}{h}(-2i\operatorname{Im}\Psi(z,w)-K(z,w))} & a_2\overline{a}_2\mathrm{e}^{-\frac{2}{h}K(z,w)}\end{pmatrix}\\ &+\frac{2}{h}\begin{pmatrix}\Psi''_{z\overline{w}}(z,z;h) & \Psi''_{z\overline{w}}(z,w;h)\mathrm{e}^{\frac{1}{h}(2i\operatorname{Im}\Psi(z,w)-K(z,w))}\\ \Psi''_{z\overline{w}}(w,z;h)\mathrm{e}^{\frac{1}{h}(-2i\operatorname{Im}\Psi(z,w)-K(z,w))} & \Psi''_{z\overline{w}}(w,w;h)\end{pmatrix}\\ &+\mathcal{O}(h^\infty).\end{aligned}$$

We will give a proof of this result further below. First, we state formulae for the trace, the determinant and the permanent of Γ.

Corollary 3.3.5. *Under the assumptions of Proposition 3.3.4, we have that*

$$\begin{aligned}\operatorname{tr}\Gamma = \frac{2}{h\left(\mathrm{e}^{\frac{2}{h}K(z,w)}-1\right)}\Big[&\left(\Psi''_{z\overline{w}}(z,z;h)+\Psi''_{z\overline{w}}(w,w;h)+\mathcal{O}(h^\infty)\right)\left(\mathrm{e}^{\frac{2}{h}K(z,w)}-1\right)\\ &-2h^{-1}(|a_1|^2+|a_2|^2)\Big],\end{aligned}$$

$$\begin{aligned}\det\Gamma =& -\frac{16}{h^4\left(1-\mathrm{e}^{-\frac{2}{h}K(z,w)}\right)}\mathrm{e}^{-\frac{2}{h}K(z,w)}\Big[|a_1a_2|^2+\frac{h}{2}\Big(|a_1|^2(\partial^2_{z\overline{w}}\Psi)(w,w;h)\\ &\qquad -2\operatorname{Re}\left\{(\partial^2_{z\overline{w}}\Psi)(w,z;h)a_1\overline{a}_2\right\}+|a_2|^2(\partial^2_{z\overline{w}}\Psi)(z,z;h)\Big)\Big]\\ &+\frac{4}{h^2}\left((\partial^2_{z\overline{w}}\Psi)(z,z;h)(\partial^2_{z\overline{w}}\Psi)(w,w;h)-(\partial^2_{z\overline{w}}\Psi)(z,w;h)(\partial^2_{z\overline{w}}\Psi)(w,z;h)\mathrm{e}^{-\frac{2}{h}K(z,w)}\right)\\ &+\mathcal{O}(h^\infty)\end{aligned}$$

and that

$$\begin{aligned}\operatorname{perm}\Gamma =& \frac{16}{h^4\left(1-\mathrm{e}^{-\frac{2}{h}K(z,w)}\right)^2}\mathrm{e}^{-\frac{2}{h}K(z,w)}|a_1a_2|^2\left(1+\mathrm{e}^{-\frac{2}{h}K(z,w)}\right)\\ &-\frac{8}{h^3\left(1-\mathrm{e}^{-\frac{2}{h}K(z,w)}\right)}\mathrm{e}^{-\frac{2}{h}K(z,w)}\Big(|a_1|^2(\partial^2_{z\overline{w}}\Psi)(w,w;h)\\ &\qquad +2\operatorname{Re}\left\{(\partial^2_{z\overline{w}}\Psi)(w,z;h)a_1\overline{a}_2\right\}+|a_2|^2(\partial^2_{z\overline{w}}\Psi)(z,z;h)\Big)\\ &+\frac{4}{h^2}\left((\partial^2_{z\overline{w}}\Psi)(z,z;h)(\partial^2_{z\overline{w}}\Psi)(w,w;h)+(\partial^2_{z\overline{w}}\Psi)(z,w;h)(\partial^2_{z\overline{w}}\Psi)(w,z;h)\mathrm{e}^{-\frac{2}{h}K(z,w)}\right)\\ &+\mathcal{O}(h^\infty).\end{aligned}$$

Proof. The result follows from a direct computation using Proposition 3.3.4; for the definition of the permanent of a matrix see (3.1.8). □

We have the following bound on the trace of Γ:

Proposition 3.3.6. *Under the assumptions of Proposition 3.3.4, we have that for* $|z-w| \gg h$

$$0 < \operatorname{tr}\Gamma \leq \mathcal{O}(h^{-1}).$$

Let us turn to the proofs of the above propositions. We begin by considering a very helpful congruency transformation. In view of Proposition 3.2.1, we prove

Lemma 3.3.7. *Let* $\Omega \Subset \Sigma$ *be as in* (1.3.1)*, and let* $D_\Omega(C)$*,* $\Phi(z;h)$ *and* $\Psi(z,w;h)$ *be as in Proposition 3.2.1, for* $(z,w) \in D_\Omega(C)$*. Let* Γ *be as in* (3.3.1)*. Define the matrices*

$$\widetilde{A} := \begin{pmatrix} e^{\frac{2}{h}\Psi(z,z;h)} & e^{\frac{2}{h}\Psi(z,w;h)} \\ e^{\frac{2}{h}\Psi(w,z;h)} & e^{\frac{2}{h}\Psi(w,w;h)} \end{pmatrix} \quad \text{and} \quad \Lambda := \begin{pmatrix} e^{-\frac{1}{h}\Phi(z;h)} & 0 \\ 0 & e^{-\frac{1}{h}\Phi(w;h)} \end{pmatrix},$$

$$\widetilde{B} := 2h^{-1}\begin{pmatrix} \Psi'_{\overline{w}}(z,z;h)e^{\frac{2}{h}\Psi(z,z;h)} & \Psi'_{\overline{w}}(z,w;h)e^{\frac{2}{h}\Psi(z,w;h)} \\ \Psi'_{\overline{w}}(w,z;h)e^{\frac{2}{h}\Psi(w,z;h)} & \Psi'_{\overline{w}}(w,w;h)e^{\frac{2}{h}\Psi(w,w;h)} \end{pmatrix}$$

and

$$\widetilde{C} := h^{-2}\begin{pmatrix} c(z,z;h)e^{\frac{2}{h}\Psi(z,z;h)} & c(z,w;h)e^{\frac{2}{h}\Psi(z,w;h)} \\ c(w,z;h)e^{\frac{2}{h}\Psi(w,z;h)} & c(w,w;h)e^{\frac{2}{h}\Psi(w,w;h)} \end{pmatrix}$$

with $c(z,w;h) := 4\Psi'_z(z,w;h)\Psi'_{\overline{w}}(z,w;h) + 2h\Psi''_{z\overline{w}}(z,w;h)$*. Then, we have for* $|z-w| \geq h^N/\mathcal{O}(1)$ *that*

$$\Gamma = \Lambda(\widetilde{C} - \widetilde{B}^*\widetilde{A}^{-1}\widetilde{B})\Lambda + \mathcal{O}_{\mathscr{C}^\infty}\left(h^\infty\right).$$

Proof. To abbreviate the notation, we define for $(z,w) \in D_\Omega(C)$ the following function

$$F(z,w) := e^{-\frac{1}{h}\Phi(z;h)}e^{-\frac{1}{h}\Phi(w;h)}e^{\frac{2}{h}\Psi(z,w;h)}.$$

By Proposition 3.2.1, we see that F is bounded by 1 and that all its derivatives are bounded polynomially in h^{-1}. Furthermore, the matrices A, B and C are given by

$$\begin{aligned} A(z,w) &= A_0(z,w) + \mathcal{O}_{\mathscr{C}^\infty}\left(h^\infty\right), \\ B(z,w) &= B_0(z,w) + \mathcal{O}_{\mathscr{C}^\infty}\left(h^\infty\right), \\ C(z,w) &= C_0(z,w) + \mathcal{O}_{\mathscr{C}^\infty}\left(h^\infty\right), \end{aligned}$$

where $(z,w) \in D_\Omega(C)$ and

$$A_0(z,w) = \begin{pmatrix} F(z,z) & F(z,w) \\ F(w,z) & F(w,w) \end{pmatrix},$$

and

$$B_0(z,w) = \begin{pmatrix} (\partial_{\overline{w}}F)(z,z) & (\partial_{\overline{w}}F)(z,w) \\ (\partial_{\overline{w}}F)(w,z) & (\partial_{\overline{w}}F)(w,w) \end{pmatrix},$$

and

$$C_0(z,w) = \begin{pmatrix} (\partial^2_{z\overline{w}}F)(z,z) & (\partial^2_{z\overline{w}}F)(z,w) \\ (\partial^2_{z\overline{w}}F)(w,z) & (\partial^2_{z\overline{w}}F)(w,w) \end{pmatrix}.$$

One computes that

$$\begin{aligned} (\partial_{\overline{w}}F)(z,w) = \frac{1}{h}\left[2(\partial_{\overline{w}}\Psi)(z,w;h) - (\partial_{\overline{w}})\Phi(w;h)\right]e^{-\frac{1}{h}\Phi(z;h)-\frac{1}{h}\Phi(w;h)}e^{\frac{2}{h}\Psi(z,w)} \\ + \mathcal{O}_{\mathscr{C}^\infty}\left(h^\infty\right), \end{aligned}$$

and that

$$
\begin{aligned}
&(\partial^2_{z\overline{w}}F)(z,w)\\
&\quad=\frac{1}{h^2}\Big[\,[2(\partial_z\Psi)(z,w;h)-(\partial_z\Phi)(z;h)]\,\big[2(\partial_{\overline{w}}\Psi)(z,w;h)-(\partial_{\overline{w}}\Phi)(w;h)\big]+\\
&\qquad\qquad 2h(\partial^2_{z\overline{w}}\Psi)(z,w;h)\Big]\mathrm{e}^{-\frac{1}{h}\Phi(z_1;h)-\frac{1}{h}\Phi(z_2;h)}\mathrm{e}^{\frac{2}{h}\Psi(z_1,z_2)}+\mathcal{O}_{\mathscr{C}^\infty}\big(h^\infty\big).
\end{aligned}
$$

Using that $\det A_0=\det A+\mathcal{O}(h^\infty)$ and that $\det A\geq h^{2N-1}/\mathcal{O}(1)$ for $|z-w|\geq h^N/\mathcal{O}(1)$ (cf. Proposition 3.3.2), we see that

$$
\Gamma=C_0-B_0^*A_0^{-1}B_0+\mathcal{O}\big(h^\infty\big).
$$

Defining,

$$
\Lambda':=\begin{pmatrix}\partial_z\mathrm{e}^{-\frac{1}{h}\Phi(z;h)} & 0\\ 0 & \partial_w\mathrm{e}^{-\frac{1}{h}\Phi(w;h)}\end{pmatrix}
$$

we see that

$$
\begin{aligned}
&A_0=\Lambda\widetilde{A}\Lambda,\\
&B_0=\Lambda(\widetilde{B})\Lambda+\Lambda\widetilde{A}(\Lambda')+\mathcal{O}_{\mathscr{C}^\infty}\big(h^\infty\big),\\
&C_0=\Lambda(\widetilde{C})\Lambda+\Lambda(\widetilde{B}^*)(\Lambda')+\Lambda'(\widetilde{B})\Lambda+\Lambda'\widetilde{A}(\Lambda')+\mathcal{O}_{\mathscr{C}^\infty}\big(h^\infty\big).
\end{aligned}
$$

A direct computation then yields that

$$
\Gamma=\Lambda(\widetilde{C}-\widetilde{B}^*\widetilde{A}^{-1}\widetilde{B})\Lambda+\mathcal{O}_{\mathscr{C}^\infty}\big((\det A)^{-1}h^\infty\big).
$$

□

Proof of Proposition 3.3.4. In view of Lemma 3.3.7, it remains to consider the matrix

$$
\widetilde{\Gamma}:=\widetilde{C}-\widetilde{B}^*\widetilde{A}^{-1}\widetilde{B}.
$$

In the sequel we will suppress the h-dependency of the function Ψ to abbreviate our notation. Recall the definition of $\widetilde{A}$ from Lemma 3.3.7 and note that

$$
\begin{aligned}
\det\widetilde{A}&=\mathrm{e}^{\frac{2}{h}\Psi(z,z)}\mathrm{e}^{\frac{2}{h}\Psi(w,w)}-\mathrm{e}^{\frac{4}{h}\operatorname{Re}\Psi(z,w)}\\
&=\mathrm{e}^{\frac{2}{h}\Psi(z,z)}\mathrm{e}^{\frac{2}{h}\Psi(w,w)}\left(1-\mathrm{e}^{-\frac{2}{h}K(z,w)}\right).
\end{aligned}
\tag{3.3.2}
$$

For $\frac{1}{C}h^N\leq|z-w|$, Proposition 3.2.1 implies that $\det\widetilde{A}$ is positive. Hence, the inverse of $\widetilde{A}$ exists and is given by

$$
\widetilde{A}^{-1}:=\frac{1}{\det\widetilde{A}}\begin{pmatrix}\mathrm{e}^{\frac{2}{h}\Psi(w,w)} & -\mathrm{e}^{\frac{2}{h}\Psi(z,w)}\\ -\mathrm{e}^{\frac{2}{h}\Psi(w,z)} & \mathrm{e}^{\frac{2}{h}\Psi(z,z)}\end{pmatrix}.
$$

To calculate $\widetilde{B}^*$, we use Lemma 3.3.7 and the symmetries of the function $\Psi(z,w)$ given in Proposition 3.2.1. Indeed, one gets that

$$
\widetilde{B}^*:=2h^{-1}\begin{pmatrix}\Psi'_z(z,z)\mathrm{e}^{\frac{2}{h}\Psi(z,z)} & \Psi'_z(z,w)\mathrm{e}^{\frac{2}{h}\Psi(z,w)}\\ \Psi'_z(w,z)\mathrm{e}^{\frac{2}{h}\Psi(w,z)} & \Psi'_z(w,w)\mathrm{e}^{\frac{2}{h}\Psi(w,w)}\end{pmatrix}
$$

and one computes that $M:=h\widetilde{B}^*\widetilde{A}^{-1}h\widetilde{B}$ is given by

$$
M=\frac{4}{\det\widetilde{A}}\begin{pmatrix}M_{11} & M_{12}\\ M_{21} & M_{22}\end{pmatrix}
$$

with

$$
\begin{aligned}
M_{11}=&\Psi'_z(z,z)\Psi'_{\overline{w}}(z,z)\mathrm{e}^{\frac{1}{h}(4\Psi(z,z)+2\Psi(w,w))}+\big[\Psi'_z(z,w)\Psi'_{\overline{w}}(w,z)\\
&-\Psi'_z(z,w)\Psi'_{\overline{w}}(z,z)-\Psi'_z(z,z)\Psi'_{\overline{w}}(w,z)\big]\mathrm{e}^{\frac{1}{h}(2\Psi(z,z)+4\operatorname{Re}\Psi(z,w))},
\end{aligned}
$$

$$M_{12} = -\Psi'_z(z,w)\Psi'_{\overline{w}}(z,w)\mathrm{e}^{\frac{1}{h}(4\Psi(z,w)+2\Psi(w,z))} + \big[\Psi'_z(z,z)\Psi'_{\overline{w}}(z,w) + \Psi'_z(z,w)\Psi'_{\overline{w}}(w,w) - \Psi'_z(z,z)\Psi'_{\overline{w}}(w,w)\big]\mathrm{e}^{\frac{2}{h}(\Psi(z,z)+\Psi(z,w)+\Psi(w,w))},$$

and

$$M_{22} = \Psi'_z(w,w)\Psi'_{\overline{w}}(w,w)\mathrm{e}^{\frac{1}{h}(2\Psi(z,z)+4\Psi(w,w))} + \big[\Psi'_z(w,z)\Psi'_{\overline{w}}(z,w) - \Psi'_z(w,w)\Psi'_{\overline{w}}(z,w) - \Psi'_z(w,z)\Psi'_{\overline{w}}(w,w)\big]\mathrm{e}^{\frac{1}{h}(2\Psi(w,w)+4\mathrm{Re}\,\Psi(z,w))}.$$

Since the matrix M is clearly self-adjoint, one has that $M_{21} = \overline{M}_{12}$. Comparing the coefficients of M with with those of $h^2(\det\widetilde{A}/4)\widetilde{C}$ (cf. Lemma 3.3.7) and using the symmetries of Ψ (cf. Proposition 3.2.1), we see that

$$\begin{aligned} h^2\widetilde{\Gamma} = &\frac{-4}{\det\widetilde{A}}\begin{pmatrix} a_1\overline{a}_1\mathrm{e}^{\frac{1}{h}(2\Psi(z,z)+4\mathrm{Re}\,\Psi(z,w))} & a_1\overline{a}_2\mathrm{e}^{\frac{2}{h}(\Psi(z,z)+\Psi(z,w)+\Psi(w,w))} \\ a_2\overline{a}_1\mathrm{e}^{\frac{2}{h}(\Psi(z,z)+\Psi(w,z)+\Psi(w,w))} & a_2\overline{a}_2\mathrm{e}^{\frac{1}{h}(2\Psi(w,w)+4\mathrm{Re}\,\Psi(z,w))}\end{pmatrix} \\ &+2h\begin{pmatrix}\Psi''_{z\overline{w}}(z,z;h)\mathrm{e}^{\frac{2}{h}\Psi(z,z)} & \Psi''_{z\overline{w}}(z,w;h)\mathrm{e}^{\frac{2}{h}\Psi(z,w)} \\ \Psi''_{z\overline{w}}(w,z;h)\mathrm{e}^{\frac{2}{h}\Psi(w,z)} & \Psi''_{z\overline{w}}(w,w;h)\mathrm{e}^{\frac{2}{h}\Psi(w,w)}\end{pmatrix} \end{aligned} \tag{3.3.3}$$

with a_i as in the hypothesis of Proposition 3.3.4. Recall from (3.2.2) that the function $K(z,w)$ is defined by

$$-K(z,w) = 2\mathrm{Re}\,\Psi(z,w) - \Phi(z) - \Phi(w)$$

where $\Phi(z) = \Psi(z,z)$. Using (3.3.2), we find that the first matrix in (3.3.3) is equal to

$$\frac{-4}{1-\mathrm{e}^{-\frac{2}{h}K(z,w)}}\begin{pmatrix} a_1\overline{a}_1\mathrm{e}^{\frac{1}{h}(2\Psi(z,z)-2K(z,w))} & a_1\overline{a}_2\mathrm{e}^{\frac{2}{h}\Psi(z,w)} \\ a_2\overline{a}_1\mathrm{e}^{\frac{2}{h}\Psi(w,z)} & a_2\overline{a}_2\mathrm{e}^{\frac{1}{h}(2\Psi(w,w)-2K(z,w))}\end{pmatrix}.$$

It follows by Lemma 3.3.7 that

$$\Gamma = \Lambda\widetilde{\Gamma}\Lambda^* + \mathcal{O}_{\mathscr{C}^\infty}(h^\infty).$$

In the last equality we used that $\det A$ is bounded from below by a power of h; see Lemma 3.3.7. Carrying out the matrix multiplication $\Lambda\widetilde{\Gamma}\Lambda^*$ implies the statement of the proposition. □

Proof of Proposition 3.3.1. The Shur complement formula yields that the determinant of the Gramian matrix G is given by $\det G = \det A \det\Gamma$. Hence, using Proposition 3.3.2 and Corollary 3.3.5, we see that

$$\begin{aligned} \det G = &-\frac{16(1+\mathcal{O}(h^\infty))}{h^4}\mathrm{e}^{-\frac{2}{h}K(z,w)}\Big[|a_1a_2|^2 + \frac{h}{2}\big(|a_1|^2(\partial^2_{z\overline{w}}\Psi)(w,w;h) \\ &\qquad - 2\mathrm{Re}\,\{(\partial^2_{z\overline{w}}\Psi)(w,z;h)a_1\overline{a}_2\} + |a_2|^2(\partial^2_{z\overline{w}}\Psi)(z,z;h)\big)\Big] \\ &+\frac{4}{h^2}\Big((\partial^2_{z\overline{w}}\Psi)(z,z;h)(\partial^2_{z\overline{w}}\Psi)(w,w;h) - (\partial^2_{z\overline{w}}\Psi)(z,w;h)(\partial^2_{z\overline{w}}\Psi)(w,z;h)\mathrm{e}^{-\frac{2}{h}K(z,w)}\Big) \\ &\cdot\Big(1-\mathrm{e}^{-\frac{2}{h}K(z,w)} + \mathcal{O}(h^\infty)\Big) + \mathcal{O}(h^\infty). \end{aligned} \tag{3.3.4}$$

Next, we consider the Taylor expansion of the terms a_1 and a_2 up to first order. Similarly as in Proposition 3.2.1, we develop around the point $(\frac{z+w}{2},\frac{z+w}{2})$ and get that

$$\begin{aligned} a_1 &= (\partial_z\Psi)(z,z) - (\partial_z\Psi)(z,w) \\ &= (\partial^2_{z\tilde{w}}\Psi)\Big(\frac{z+w}{2},\frac{z+w}{2}\Big)(z-w) + \mathcal{O}(|z-w|^2+h^\infty) \end{aligned} \tag{3.3.5}$$

and

$$\begin{aligned} a_2 &= (\partial_z\Psi)(w,z) - (\partial_z\Psi)(w,w) \\ &= (\partial^2_{z\tilde{w}}\Psi)\Big(\frac{z+w}{2},\frac{z+w}{2}\Big)(z-w) + \mathcal{O}(|z-w|^2+h^\infty). \end{aligned} \tag{3.3.6}$$

Moreover, one has that for $\zeta,\omega\in\{z,w\}$

$$(\partial^2_{z\bar w}\Psi)(\zeta,\omega)=(\partial^2_{z\bar w}\Psi)\left(\frac{z+w}{2},\frac{z+w}{2}\right)+\mathcal{O}(|z-w|+h^\infty). \tag{3.3.7}$$

Since we suppose that $|z-w|\gg h^{3/5}$, the above error term is equal to $\mathcal{O}(|z-w|)$. Since $\partial^2_{z\bar w}\Psi$ is evaluated at a point on the diagonal, it follows from Proposition 3.2.1, that

$$\begin{aligned}(\partial^2_{z\bar w}\Psi)\left(\frac{z+w}{2},\frac{z+w}{2}\right)&=(\partial^2_{z\bar z}\Phi)\left(\frac{z+w}{2},\frac{z+w}{2}\right)\\&=\frac{1}{4}\sigma\left(\frac{z+w}{2}\right)+\mathcal{O}(h)=:\frac{1}{4}\sigma_h(z,w).\end{aligned} \tag{3.3.8}$$

Plugging the above Taylor expansion into (3.3.4), one gets that $\det G$ is equal to

$$\begin{aligned}&\frac{\sigma_h(z,w)^2}{4h^2}\Big\{\Big[1+\mathcal{O}(|z-w|)-(1+\mathcal{O}(|z-w|))\mathrm{e}^{-\frac{2}{h}K(z,w)}\Big]\Big(1-\mathrm{e}^{-\frac{2}{h}K(z,w)}+\mathcal{O}(h^\infty)\Big)\\&\quad-4\mathrm{e}^{-\frac{2}{h}K(z,w)}\left(\left(\frac{\sigma_h(z,w)|z-w|^2}{4h}\right)^2(1+\mathcal{O}(|z-w|))+\frac{\sigma_h(z,w)|z-w|^2}{4h}\mathcal{O}(|z-w|)\right)\Big\}\\&\quad+\mathcal{O}(h^\infty)\\&=\frac{\sigma_h(z,w)^2}{4h^2}\Big\{\Big(1-\mathrm{e}^{-\frac{2}{h}K(z,w)}\Big)^2+\mathcal{O}(|z-w|)\Big(1-\mathrm{e}^{-\frac{2}{h}K(z,w)}\Big)+\mathcal{O}(h^\infty)\\&\quad-4\mathrm{e}^{-\frac{2}{h}K(z,w)}\left[\left(\frac{\sigma_h(z,w)|z-w|^2}{4h}\right)^2+\mathcal{O}\left(\frac{|z-w|^5}{h^2}\right)+\mathcal{O}\left(\frac{|z-w|^3}{h}\right)\right]\Big\}.\end{aligned}$$

Recall from (3.2.2) that $K(z,w)\asymp|z-w|^2$, wherefore we see that $\det G$ is positive for $|z-w|\gg\sqrt{h}$. Next, we suppose that $|z-w|\asymp\sqrt{h}$. Hence, one gets that

$$\begin{aligned}\det G=\frac{\sigma_h(z,w)^2\mathrm{e}^{-\frac{2}{h}K(z,w)}}{h^2}&\Big\{\sinh^2\frac{K(z,w)}{h}+\mathcal{O}(|z-w|)\Big(\mathrm{e}^{\frac{2}{h}K(z,w)}-1\Big)+\mathcal{O}(h^\infty)\\&-\left[\left(\frac{\sigma_h(z,w)|z-w|^2}{4h}\right)^2+\mathcal{O}\left(\frac{|z-w|^5}{h^2}\right)+\mathcal{O}\left(\frac{|z-w|^3}{h}\right)\right]\Big\}.\end{aligned} \tag{3.3.9}$$

Using the Taylor expansion of the $\sinh x$ and (3.2.2), one gets that

$$\begin{aligned}&\sinh^2\frac{K(z,w)}{h}-\left(\frac{\sigma_h(z,w)|z-w|^2}{4h}\right)^2\\&\quad\geq\left(\frac{1}{3}\frac{\sigma_h(z,w)|z-w|^2}{4h}\right)^4(1+\mathcal{O}(|z-w|))+\mathcal{O}\left(\frac{\sigma_h(z,w)|z-w|^5}{h^2}\right).\end{aligned} \tag{3.3.10}$$

Note that the principal term on the right hand side of the inequality dominates the error terms. The same holds true for the other error terms in (3.3.9).

Next, let us suppose that $h^{3/5}\ll|z-w|\ll\sqrt{h}$. Since

$$\mathcal{O}(|z-w|)\Big(\mathrm{e}^{\frac{2}{h}K(z,w)}-1\Big)=\mathcal{O}\left(\frac{|z-w|^3}{h}\right),$$

it follows by (3.3.9) and (3.3.10) that $\det G$ is positive for $|z-w|\gg h^{3/5}$. □

Proof of Proposition 3.3.6. Using (3.3.5), (3.3.6) and (3.3.7), one gets that

$$\begin{aligned}\operatorname{tr}\Gamma=\frac{\sigma_h(z,w)}{2h\Big(\mathrm{e}^{\frac{2}{h}K(z,w)}-1\Big)}&\Big[\Big(\mathrm{e}^{\frac{2}{h}K(z,w)}-1\Big)(1+\mathcal{O}(|z-w|))\\&-\frac{\sigma_h(z,w)|z-w|^2}{2h}(1+\mathcal{O}(|z-w|))\Big].\end{aligned} \tag{3.3.11}$$

Since

$$\mathrm{e}^{\frac{2}{h}K(z,w)}-1\geq\frac{\sigma_h(z,w)|z-w|^2}{2h}(1+\mathcal{O}(|z-w|))+\frac{\sigma_h(z,w)|z-w|^4}{8h^2}(1+\mathcal{O}(|z-w|)),$$

it follows that for $|z-w|\gg h$ the trace of Γ is positive. Furthermore, the above inequality applied to (3.3.11), implies the upper bound stated in the Proposition. □

3.3.3 – The permanent of Γ

The permanent of the matrix Γ (cf. (3.3.1)) is vital to the 2-point density of eigenvalues and therefore, we shall give a more detailed description of it than the one given in Corollary 3.3.5.

Proposition 3.3.8. *Let $\sigma_h(z,w)$ be as in Theorem 1.3.4 and let $K(z,w)$ be as in (3.2.2). Under the assumptions of Proposition 3.3.4, we have that for $N>1$ and $\frac{1}{C}h^N \leq |z-w|$,*

$$\begin{aligned}
&\operatorname{perm}\Gamma(z,w;h)\\
&= \frac{1}{4h^2}\Bigg[\sigma_h(z,z)\sigma_h(w,w)+\sigma_h(z,w)^2(1+\mathcal{O}(|z-w|))\mathrm{e}^{-\frac{2K(z,w)}{h}}+\mathcal{O}(h^\infty)\\
&+\frac{\sigma_h(z,w)^2(1+\mathcal{O}(|z-w|))}{\mathrm{e}^{\frac{K(z,w)}{h}}\sinh\frac{K(z,w)}{h}}\left(\left(\frac{\sigma_h(z,w)|z-w|^2}{4h}\right)^2 2\coth\frac{K(z,w)}{h}-\frac{\sigma_h(z,w)|z-w|^2}{h}\right)\Bigg].
\end{aligned}$$

Proof. Applying (3.3.5), (3.3.6) and (3.3.7) to the formula for $\operatorname{perm}\Gamma$ given in Proposition 3.3.6 and using the notation introduced in (3.3.8), one gets that

$$\begin{aligned}
\operatorname{perm}\Gamma =&\frac{8\coth\frac{K}{h}}{h^4\sinh\frac{K}{h}}\mathrm{e}^{-\frac{1}{h}K(z,w)}|4^{-2}\sigma_h(z,w)^2(z-w)^2(1+\mathcal{O}(|z-w|)|^2\\
&-\frac{\mathrm{e}^{-\frac{1}{h}K(z,w)}}{4h^3\sinh\frac{K}{h}}\sigma_h(z,w)^3|z-w|^2(1+\mathcal{O}(|z-w|)\\
&+\frac{1}{4h^2}\left(\sigma_h(z,z)\sigma_h(w,w)+\sigma_h(z,w;h)^2(1+\mathcal{O}(|z-w|)\mathrm{e}^{-\frac{2}{h}K(z,w)}\right)\\
&+\mathcal{O}(h^\infty).
\end{aligned}$$

Thus, one computes that

$$\begin{aligned}
\operatorname{perm}\Gamma =&\frac{\sigma_h(z,w)^2(1+\mathcal{O}(|z-w|)}{4h^2\mathrm{e}^{\frac{1}{h}K(z,w)}\sinh\frac{K}{h}}\left[\left(\frac{\sigma_h(z,w)|z-w|^2}{4h}\right)^2 2\coth\frac{K}{h}-\frac{\sigma_h(z,w)|z-w|^2}{h}\right]\\
&+\frac{1}{4h^2}\left(\sigma_h(z,z)\sigma_h(w,w)+\sigma_h(z,w;h)^2(1+\mathcal{O}(|z-w|)\mathrm{e}^{-\frac{2}{h}K(z,w)}\right)\\
&+\mathcal{O}(h^\infty)
\end{aligned}$$

and we conclude the statement of the proposition. □

3.4 | Proof of the results on the eigenvalue interaction

We begin by proving the results of Theorem 1.3.4, Proposition 1.3.5 and of Proposition 1.3.6.

Proof of Theorem 1.3.4. The result follows directly from Proposition 3.1.7 with the density D given by Proposition 3.3.8 and by Proposition 3.3.2. □

Proof of Proposition 1.3.5. First, let us treat the case of the long range interaction: we suppose that $|z-w|\gg(h\ln h^{-1})^{\frac{1}{2}}$. Here, we have that for any power $N>1$ the term

$$\left(\frac{\sigma_h(z,w)|z-w|^2}{4h}\right)^N\mathrm{e}^{-K(z,w)}$$

remains bounded. Using that $\sinh K(z,w)\geq\mathcal{O}(h^{-C})>0$ with $C\gg1$ and using that $\sigma_h(z,z)=\sigma(z)+\mathcal{O}(h)$, it follows that

$$D^\delta(z,w;h)=\frac{\sigma(z)\sigma(w)+\mathcal{O}(h)}{(2h\pi)^2}\left(1+\mathcal{O}\left(\delta h^{-\frac{8}{5}}\right)\right).$$

Next, we consider the case where $h^{\frac{4}{7}} \ll |z-w| \ll h^{\frac{1}{2}}$. Recall from Theorem 1.3.4 that

$$D^{\delta}(z,w;h) = \frac{\Lambda(z,w)}{(2\pi h)^2\left(1-\mathrm{e}^{-2K(z,w)}\right)}\left(1+\mathcal{O}\left(\delta h^{-\frac{8}{5}}\right)\right)+\mathcal{O}\left(\mathrm{e}^{-\frac{D}{h^2}}\right) \tag{3.4.1}$$

$$D^{\delta}(z,w;h) = \Lambda(z,w)\left(1+\mathcal{O}\left(h^{\infty}+\delta h^{-\frac{51}{10}}\right)\right)+\mathcal{O}\left(\mathrm{e}^{-\frac{D}{h^2}}\right),$$

with $\Lambda(z,w;h)$ equal to

$$\begin{aligned}&\sigma_h(z,z)\sigma_h(w,w)+\sigma_h(z,w)^2(1+\mathcal{O}(|z-w|))\mathrm{e}^{-2K(z,w)}+\mathcal{O}\left(h^{\infty}+\delta h^{-\frac{31}{10}}\right)\\ &+\frac{\sigma_h(z,w)^2(1+\mathcal{O}(|z-w|))}{\mathrm{e}^{K(z,w)}\sinh K(z,w)}\left(\left(\frac{\sigma_h(z,w)|z-w|^2}{4h}\right)^2 2\coth K(z,w)-\frac{\sigma_h(z,w)|z-w|^2}{h}\right).\end{aligned}$$

Similarly to (3.3.7), we have that $\sigma_h(z,z)=\sigma_h(z,w)(1+\mathcal{O}(|z-w|)$. We start by considering the first term in (3.4.1):

$$\frac{\Lambda(z,w)}{(2\pi h)^2\left(1-\mathrm{e}^{-2K(z,w)}\right)}. \tag{3.4.2}$$

Set $\sigma_h=\sigma_h(z,w)$. Using the Taylor expansions of the functions $\sinh x$, $\coth x$ and e^{-x}, one computes, that (3.4.2) is equal to

$$\begin{aligned}&\frac{1}{h\pi^2\sigma_h|z-w|^2\left(1+\mathcal{O}\left(\frac{|z-w|^2}{h}\right)\right)}\Bigg[\sigma_h^2(1+\mathcal{O}(|z-w|))-\frac{\sigma_h^3|z-w|^2}{4h}(1+\mathcal{O}(|z-w|))\\ &+\frac{\sigma_h^4|z-w|^4}{4^2h^2}\left(1+\mathcal{O}\left(\frac{|z-w|^2}{h}\right)\right)+\left\{\frac{\sigma_h^4|z-w|^4}{3\cdot 4^4h^2}\left(1+\mathcal{O}\left(\frac{|z-w|^4}{h^2}\right)\right)-1\right\}\cdot\\ &\cdot\frac{\sigma_h^2\left(1-\frac{\sigma_h|z-w|^2}{4h}(1+\mathcal{O}(|z-w|))+\frac{\sigma_h^2|z-w|^4}{2\cdot 4^2h}\left(1+\mathcal{O}\left(\frac{|z-w|^2}{h}\right)\right)\right)}{1+\mathcal{O}(|z-w|)+\frac{\sigma_h^2|z-w|^4}{4^2\cdot 6h}\left(1+\mathcal{O}\left(\frac{|z-w|^2}{h}\right)\right)}+\mathcal{O}\left(h^{\infty}+\delta h^{-\frac{31}{10}}\right)\Bigg]\end{aligned}$$

which simplifies to

$$\Lambda(z,w;h)=\frac{\sigma_h^3|z-w|^2}{(4\pi h)^2}\left(1+\mathcal{O}\left(\frac{|z-w|^2}{h}\right)\right).$$

Hence,

$$D^{\delta}(z,w;h)=\frac{\sigma_h^3|z-w|^2}{(4\pi h)^2}\left(1+\mathcal{O}\left(\frac{|z-w|^2}{h}+\delta h^{-\frac{8}{5}}\right)\right)$$

which concludes the proof. □

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Proof of Proposition 1.3.6. Using that $\sigma_h(z,w_0)=\sigma_h(z,z)(1+\mathcal{O}(|z-w_0|)$ (cf. (3.3.7) and (3.3.8)), the result of Proposition 1.3.6 follows from Proposition 1.3.5. □

It remains to prove Proposition 3.1.7. However, first, we state a global version of the implicit function theorem.

Lemma 3.4.1. *Let $0<R_0<R$, let $n,m\in\mathbb{N}$, with $n>m$, and let $B(0,R)\subset\mathbb{C}^n=\mathbb{C}_z^{n-m}\times\mathbb{C}_w^m$ denote the complex open ball of radius $R>0$ centered at 0. For $z\in B_{\mathbb{C}^{n-m}}(0,R_0)$, define $R(z):=(R^2-\|z\|^2_{\mathbb{C}^{n-m}})^{1/2}$. We consider a holomorphic function*

$$F: B(0,R)\longrightarrow\mathbb{C}^m$$

such that

- *for all* $(z,w) \in B(0,R)$ *the Jacobian of* F *with respect to* w *is given by*
$$\frac{\partial F(z,w)}{\partial w} = A + G(z,w),$$
where $G : B(0,R) \longrightarrow \mathbb{C}^{m\times m}$ *is a matrix-valued holomorphic function and*

- $A \in \mathrm{GL}_m(\mathbb{C})$ *such that*
$$\|A^{-1}\| \cdot \|G(z,w)\| \le \theta < 1$$
for all $(z,w) \in B(0,R)$.

Then, for all $z \in B_{\mathbb{C}^{n-m}}(0,R_0)$ *and for all* $y \in B_{\mathbb{C}^m}(F(z,0), \frac{1-\theta}{\|A^{-1}\|} r)$*, with* $0 < r < R(z)$*, the equation*
$$F(z,w) = y \tag{3.4.3}$$
has exactly one solution $w(z,y) \in B_{\mathbb{C}^m}(0,R(z))$*, it satisfies* $w(z,y) \in B_{\mathbb{C}^m}(0,r)$ *and it depends holomorphically on* z *and on* y.

Remark 3.4.2. Observe that the choice of $R_0 < R$ yields a uniform lower bound on $R(z)$ and so we can choose the radius of the ball $B_{\mathbb{C}^m}(F(z,0), \frac{1-\theta}{\|A^{-1}\|} r)$ uniformly in z. This will become important in the proof of Proposition 3.1.7.

Proof. Let $z \in B_{\mathbb{C}^{n-m}}(0,R_0)$ and set
$$B_{\mathbb{C}^m}(0,R(z)) \ni w \longmapsto \widetilde{F}(w) := F(z,w).$$
We begin by observing that $d\widetilde{F}(w)$ is invertible for all $w \in B_{\mathbb{C}^m}(0,R(z))$ and the norm of the inverse is bounded (uniformly in z). Indeed, for one has that
$$\left\| \left(d\widetilde{F}(w)\right)^{-1} \right\| \le \|A^{-1}\| \cdot \|(1 + A^{-1}G(z,w))^{-1}\| \le \frac{\|A^{-1}\|}{1-\theta}.$$

Claim #1: $\widetilde{F}$ is injective.

Let $w_0, w_1 \in B_{\mathbb{C}^m}(0,R(z))$ and define $y_i := \widetilde{F}(w_i)$. Hence, with $w_t := (1-t)w_0 + t w_1$, we have that
$$\frac{d}{dt}\widetilde{F}(w_t) = d\widetilde{F}(w_t) \cdot (w_1 - w_0) = (A + G(z,w_t)) \cdot (w_1 - w_0).$$
Thus,
$$y_1 - y_0 = (A + H(z,w_1,w_0)) \cdot (w_1 - w_0), \quad H(z,w_1,w_0)) = \int_0^1 G(z,w_t)\,dt,$$
where $\|H(z,w_1,w_0)\| \le \sup_{B(0,R)} \|G(z,w)\|$. Therefore, $\|A^{-1}\| \cdot \|H(z,w_1,w_0)\| \le \theta < 1$, and we see that $(A + H(z,w_1,w_0))$ is invertible and the norm of its inverse is $\le \frac{\|A^{-1}\|}{1-\theta}$ (uniformly in z). Hence,
$$\|w_1 - w_0\| \le \frac{\|A^{-1}\|}{1-\theta} \|y_1 - y_0\|, \tag{3.4.4}$$
and we conclude that $\widetilde{F}$ is injective. In particular, we have proven the uniqueness of the solution to the equation (3.4.3).

Claim #2: Let $0 < r < R(z)$. Then, for all $y \in B_{\mathbb{C}^m}(\widetilde{F}(0), \frac{1-\theta}{\|A^{-1}\|} r)$ there exists a $w \in B_{\mathbb{C}^m}(0,r)$ such that
$$\widetilde{F}(w) = y.$$
For $y = \widetilde{F}(0)$, we take $w = 0$. Using the fact that $d\widetilde{F}$ is invertible everywhere, the implicit function theorem implies that for all $y \in B(\widetilde{F}(0),\rho)$ there exists a solution $w \in B_{\mathbb{C}^m}(0,r)$, if $\rho > 0$ is small enough (cf. (3.4.4)). Let $y \in B_{\mathbb{C}^m}(\widetilde{F}(0), \frac{1-\theta}{\|A^{-1}\|} r)$, and define $y_t := (1-t)\widetilde{F}(0) + t y$. Let $t_0 \in [0,1]$ be the supremum of $\widetilde{t} \in [0,1]$ such that there exists a solution to $\widetilde{F}(w_t) = y_t$ for all $0 \le t \le \widetilde{t}$.

We have already proven that $t_0 > 0$. As $t \nearrow t_0$ we have that $w_t \in B_{\mathbb{C}^m}(0,r)$. Since $B_{\mathbb{C}^m}(0,r)$ is relatively compact in $B_{\mathbb{C}^m}(0,R(z))$, there exists a sequence $t_j \nearrow t_0$ such that $w_{t_j} \to \widetilde{w}$ with $\widetilde{w} \in \overline{B_{\mathbb{C}^m}(0,r)}$. Thus,

$$\widetilde{F}(\widetilde{w}) = y_{t_0},$$

and we see by (3.4.4) that $\widetilde{w} \in B_{\mathbb{C}^m}(0,r)$.

If $t_0 < 1$, we get by the implicit function theorem, that for all $y \in B(y_{t_0}, \delta)$, with $\delta > 0$ small enough, there exists a solution $w \in B_{\mathbb{C}^m}(0,r)$. Therefore, we can solve $\widetilde{F}(w_t) = y_t$ for all $0 < t < t_0 + \delta$, which is a contradiction. Hence, $t_0 = 1$, which concludes the proof of the existence of a solution.

Finally, note that for all $(z,w) \in B(0,R)$ the Jacobian $\partial F(z,w)/\partial w$ is invertible and the norm of its inverse is uniformly bounded, indeed

$$\left\| \left(\frac{\partial F(z,w)}{\partial w} \right)^{-1} \right\| \leq \|A^{-1}\| \cdot \|(1 + A^{-1}G(z,w))^{-1}\| \leq \frac{\|A^{-1}\|}{1-\theta}.$$

In particular, we have that the determinant of the Jacobian is never equal to 0, and we conclude by the holomorphic implicit function theorem that the solution $w(z,y)$ to the equation (3.4.4) depends holomorphically on z and y. □

Proof of Proposition 3.1.7. In view of (3.1.4), it remains to study the integral

$$I(z_1,z_2,h) = \lim_{\varepsilon \to 0^+} \pi^{-N} \int_{B(0,R)} H_\varepsilon^\delta(z_1,z_2,\alpha;h) \mathrm{e}^{-\alpha\overline{\alpha}} L(d\alpha). \tag{3.4.5}$$

with

$$H_\varepsilon^\delta(z_1,z_2,\alpha;h) := \prod_{k=1}^{2} \varepsilon^{-2} \chi \left(\frac{E_{-+}^\delta(z_k,\alpha)}{\varepsilon} \right) |\partial_{z_k} E_{-+}^\delta(z_k,\alpha)|^2$$

for $1/C \geq |z_1 - z_2| \gg h^{3/5}$. We begin by performing a change of variables in the α-space.

Change of variables: For $X(z) \in \mathbb{C}^N$ as in Definition 3.1.2, define the matrix

$${}^tV := \big(X(z_1), X(z_2), \partial_{z_1} X(z_1), \partial_{z_2} X(z_2)\big) \in \mathbb{C}^{N\times 4}$$

and note that the Gramian matrix G (cf. (3.1.6)) satisfies

$$G = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} = V \cdot V^*.$$

Moreover, G is invertible by virtue of Proposition 3.3.1, since $|z_1 - z_2| \gg h^{3/5}$. Next, we define the matrix $U \in \mathbb{C}^{4\times 4}$ by

$$U := \begin{pmatrix} 1 & 0 \\ B^*A^{-1} & 1 \end{pmatrix}.$$

U is invertible and thus satisfies that $(U^{-1})^* = (U^*)^{-1}$. Define the matrix

$$\widetilde{G} := \begin{pmatrix} A & 0 \\ 0 & \Gamma \end{pmatrix} \in \mathbb{C}^{4\times 4},$$

and notice that

$$U \begin{pmatrix} A & 0 \\ 0 & \Gamma \end{pmatrix} U^* = \begin{pmatrix} 1 & 0 \\ B^*A^{-1} & 1 \end{pmatrix} \widetilde{G} \begin{pmatrix} 1 & A^{-1}B \\ 0 & 1 \end{pmatrix} = G.$$

We see that $\widetilde{G} = U^{-1}G(U^*)^{-1}$. Next, we define the matrix

$$\widetilde{V}^* := (U^{-1}V)^* \widetilde{G}^{-\frac{1}{2}} \in \mathbb{C}^{N\times 4}. \tag{3.4.6}$$

$\widetilde{V}^*$ is an isometry since $\widetilde{V}\widetilde{V}^* = 1_{\mathbb{C}^4}$. Thus, its columns form an orthonormal family in $\mathbb{C}^N$. It follows from (3.4.6) that the kernel of V and of $\widetilde{V}$ are equal, i.e. $\mathcal{N}(V) = \mathcal{N}(\widetilde{V})$. The same holds true for the range of $\widetilde{V}$ and of V, i.e. $\mathcal{R}(V) = \mathcal{R}(\widetilde{V})$.

Next, we choose an orthonormal basis, $e_1, \dots, e_N \in \mathbb{C}^N$, of the space of random variables α such that $\widetilde{V}_1^*, \dots, \widetilde{V}_4^*$, the column vectors of the matrix $\widetilde{V}^*$, are among them. In particular, let $e_i = \widetilde{V}_i^*$ for $i = 1, \dots, 4$, and let $e_5, \dots, e_N$ be in the orthogonal complement of the space spanned by $e_1, \dots, e_4$. Hence, we write for $\alpha \in \mathbb{C}^N$

$$\alpha = \sum_{i=1}^{N} \widetilde{\alpha}_i e_i,$$

where $\widetilde{\alpha} = (\widetilde{\alpha}_1, \dots, \widetilde{\alpha}_N) \in \mathbb{C}^N$. Moreover, note that

$$\alpha^* \cdot \alpha = \widetilde{\alpha}^* \cdot \widetilde{\alpha}. \tag{3.4.7}$$

Remark 3.4.3. The fact that we can only guarantee the invertibility of G for $h^{\frac{3}{5}} \ll |z-w| \ll 1$ makes (1.3.4) necessary. This might be avoided by choosing another set of basis vectors.

Next, we apply this change of variables to the vector F given in (3.1.5) and we get

$$\begin{aligned}
&F(z, \alpha(\widetilde{\alpha}); \delta, h) \\
&= \begin{pmatrix} E_{-+}(z_1) \\ E_{-+}(z_2) \\ (\partial_z E_{-+})(z_1) \\ (\partial_z E_{-+})(z_2) \end{pmatrix} - \delta \begin{pmatrix} {}^tX(z_1) \\ {}^tX(z_2) \\ {}^t(\partial_z X)(z_1) \\ {}^t(\partial_z X)(z_2) \end{pmatrix} \cdot \alpha(\widetilde{\alpha}) + \begin{pmatrix} T(z_1, \alpha(\widetilde{\alpha})) \\ T(z_2, \alpha(\widetilde{\alpha})) \\ (\partial_z T)(z_1, \alpha(\widetilde{\alpha})) \\ (\partial_z T)(z_2, \alpha(\widetilde{\alpha})) \end{pmatrix} \\
&= \begin{pmatrix} E_{-+}(z_1) \\ E_{-+}(z_2) \\ (\partial_z E_{-+})(z_1) \\ (\partial_z E_{-+})(z_2) \end{pmatrix} - \delta (V \cdot \widetilde{V}) \cdot \begin{pmatrix} \widetilde{\alpha}_1 \\ \vdots \\ \widetilde{\alpha}_4 \end{pmatrix} + \begin{pmatrix} T(z_1, \alpha(\widetilde{\alpha})) \\ T(z_2, \alpha(\widetilde{\alpha})) \\ (\partial_z T)(z_1, \alpha(\widetilde{\alpha})) \\ (\partial_z T)(z_2, \alpha(\widetilde{\alpha})) \end{pmatrix}.
\end{aligned}$$

Furthermore, one computes that

$$V\widetilde{V} = U\widetilde{G}^{\frac{1}{2}} = \begin{pmatrix} A^{\frac{1}{2}} & 0 \\ B^* A^{-\frac{1}{2}} & \Gamma^{\frac{1}{2}} \end{pmatrix}, \tag{3.4.8}$$

and we get that

$$F(z, \alpha(\widetilde{\alpha}); \delta, h) = \begin{pmatrix} E_{-+}(z_1) \\ E_{-+}(z_2) \\ (\partial_z E_{-+})(z_1) \\ (\partial_z E_{-+})(z_2) \end{pmatrix} - \delta U\widetilde{G}^{\frac{1}{2}} \cdot \begin{pmatrix} \widetilde{\alpha}_1 \\ \vdots \\ \widetilde{\alpha}_4 \end{pmatrix} + \begin{pmatrix} T(z_1, \alpha(\widetilde{\alpha})) \\ T(z_2, \alpha(\widetilde{\alpha})) \\ (\partial_z T)(z_1, \alpha(\widetilde{\alpha})) \\ (\partial_z T)(z_2, \alpha(\widetilde{\alpha})) \end{pmatrix}.$$

Next, to simplify our notation, we call the $\widetilde{\alpha}$ variables again α. Also, to abbreviate our notation, define

$$\mu(z, w; h) := \begin{pmatrix} E_{-+}(z_1) \\ E_{-+}(z_2) \end{pmatrix} \text{ and } \tau(z, \alpha; h, \delta) := \begin{pmatrix} T(z_1, \alpha) \\ T(z_2, \alpha) \end{pmatrix}.$$

and

$$\partial_z \mu(z, w; h) := \begin{pmatrix} (\partial_z E_{-+})(z_1) \\ (\partial_z E_{-+})(z_2) \end{pmatrix} \text{ and } \partial_z \tau(z, \alpha; h, \delta) := \begin{pmatrix} (\partial_z T)(z_1, \alpha) \\ (\partial_z T)(z_2, \alpha) \end{pmatrix}.$$

Remark 3.4.4. Recall that T (cf. (3.1.3)) depends on h and on δ, though not explicit in the above notation.

When we write $\partial_z \mu$ and $\partial_z \tau$ the derivatives are to be understood component wise, each of which only depends either on z_1 or z_2.

Hence,

$$F^{\delta}(z,\alpha) := F(z,\alpha;\delta,h) = \begin{pmatrix} \mu(z,h,\delta) \\ \partial_z \mu(z,h,\delta) \end{pmatrix} - \delta U \widetilde{G}^{\frac{1}{2}} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_4 \end{pmatrix} + \begin{pmatrix} \tau(z,\alpha,h,\delta) \\ \partial_z \tau(z,\alpha,h,\delta) \end{pmatrix}. \tag{3.4.9}$$

As noted in Remark 3.1.4, μ and τ are smooth in z, and τ is holomorphic in α. Moreover, τ satisfies the estimates

$$\tau_i = \mathcal{O}\left(h^{-5/2}\delta^2\right),\ i = 1,2 \text{ and } \partial_{z_i}\tau_i = \mathcal{O}\left(h^{-7/2}\delta^2\right),\ i = 1,2; \tag{3.4.10}$$

and μ satisfies the estimates

$$\mu_i = \mathcal{O}\left(h^{1/2}\mathrm{e}^{-\frac{S}{h}}\right),\ \ \partial_{z_i}\mu_i = \mathcal{O}\left(h^{-1/2}\mathrm{e}^{-\frac{S}{h}}\right),\ i = 1,2 \tag{3.4.11}$$

with S as in Definition 1.2.2. Finally, we perform the above described change of variables in the integral (3.4.5), and, using the fact that we chose an orthonormal basis of the α-space, we get that

$$H_{\varepsilon}^{\delta}(z_1,z_2,\alpha;h) = \prod_{k=1}^{2} \varepsilon^{-2}\chi\left(\frac{F_k^{\delta}(z_k,\alpha)}{\varepsilon}\right)|F_{k+2}^{\delta}(z_k,\alpha)|^2.$$

Next, let $\alpha = (\alpha_1,\alpha_2,\alpha') = (\widetilde{\alpha},\alpha')$ and split the ball $B(0,R)$, $R = Ch^{-1}$, into two pieces: pick $C_0 > 0$ such that $0 < C_1 < C_0 < C < 2C_0$, and define $R_0 = C_0 h^{-1}$. Then, we perform the splitting: $I(z,h) = I_1(z,h) + I_2(z,h)$ with

$$I_1(z,h) := \lim_{\varepsilon\to 0^+} \pi^{-N} \int\limits_{\substack{B(0,R) \\ \|\alpha'\|_{\mathbb{C}^{N-2}} \le R_0}} H_{\varepsilon}^{\delta}(z_1,z_2,\alpha;h)\mathrm{e}^{-\alpha^*\alpha} L(d\alpha).$$

and

$$I_2(z,h) := \lim_{\varepsilon\to 0^+} \pi^{-N} \int\limits_{\substack{B(0,R) \\ R_0 < \|\alpha'\|_{\mathbb{C}^{N-2}} < R}} H_{\varepsilon}^{\delta}(z_1,z_2,\alpha;h)\mathrm{e}^{-\alpha^*\alpha} L(d\alpha). \tag{3.4.12}$$

The integral I_1 First, we perform a new change of variables in the α-space. Let $\beta_1,\dots,\beta_N \in \mathbb{C}$ such that

$$\beta_1 = F_1^{\delta}(z_1,\alpha),\ \beta_2 = F_2^{\delta}(z_2,\alpha) \text{ and } \beta_i = \alpha_i, \text{ for } i = 3,\dots,N.$$

We use the following notation: $\beta = (\beta_1,\beta_2,\beta') = (\widetilde{\beta},\alpha')$. It is sufficient to check that we can express $\widetilde{\alpha} = (\alpha_1,\alpha_2)$ as a function of $(\widetilde{\beta},\alpha')$. Therefore, we apply Lemma 3.4.1 to the function

$$\mathscr{F}^{\delta}(z,\alpha) = \begin{pmatrix} F_1^{\delta}(z_1,\alpha) \\ F_2^{\delta}(z_2,\alpha) \end{pmatrix}.$$

where α plays the role of (z,w) in the Lemma. In particular, $\widetilde{\alpha}$ plays the role of w. Let us check that the assumptions of Lemma 3.4.1 are satisfied: $\mathscr{F}^{\delta}(z,\alpha)$ is by definition holomorphic in α. Using (3.4.9) and (3.4.8), we see that its Jacobian, with respect to the variables $\widetilde{\alpha}$, is given by

$$\frac{\partial\mathscr{F}(z,\alpha)}{\partial\widetilde{\alpha}} = \frac{\partial\tau}{\partial\widetilde{\alpha}} - \delta A^{\frac{1}{2}} \tag{3.4.13}$$

The Cauchy inequalities and (3.4.10) imply that

$$\frac{\partial\tau_i}{\partial\widetilde{\alpha}_j} = \mathcal{O}\left(\delta^2 h^{-\frac{3}{2}}\right), \quad i,j = 1,2.$$

This estimate is uniform in $\alpha \in B(0,R)$ and $(z_1,z_2) \in \operatorname{supp}\varphi$. Expansion of the determinant yields that

$$\det\left(\frac{\partial\tau}{\partial\widetilde{\alpha}} - \delta A^{\frac{1}{2}}\right) = \delta^2\left(\sqrt{\det A} + \mathcal{O}\left(\delta h^{-\frac{3}{2}}\right)\right). \tag{3.4.14}$$

Using that A is self-adjoint, we see by Corollary 3.3.3 that for $(z_1, z_2) \in \operatorname{supp} \varphi$

$$\|A^{-\frac{1}{2}}\| \leq \frac{1}{\min\limits_{\lambda\in\sigma(A)} \sqrt{\lambda}} \leq \mathcal{O}\left(h^{-\frac{1}{20}}\right). \tag{3.4.15}$$

By the hypothesis (1.3.2), we have that $\delta \ll h^{7/2}$. Hence, one gets that for all $\alpha \in B(0,R)$

$$\delta^{-1}\|A^{-\frac{1}{2}}\|\cdot\|\partial_{\widetilde{\alpha}}\tau\| \leq \mathcal{O}\left(\delta h^{-\frac{3}{2}-\frac{1}{20}}\right) \ll 1.$$

Hence $\mathscr{F}^{\delta}(z,\alpha)$ satisfies the assumptions of Lemma 3.4.1. In the integral I_1 we restricted α' to the open ball $\|\alpha'\|_{\mathbb{C}^{N-2}} < R_0$. It follows by Lemma 3.4.1 that for all

$$\widetilde{\beta} \in B_{\mathbb{C}^2}\left(\mathscr{F}^{\delta}(z;0,\alpha'), r\right) \tag{3.4.16}$$

with

$$\begin{aligned} r &:= \left(\delta\|A^{-\frac{1}{2}}\|^{-1}(1 - \max_{\alpha\in B(0,R)} \delta^{-1}\|A^{-\frac{1}{2}}\|\cdot\|\partial_{\widetilde{\alpha}}\tau\|)\right)\sqrt{R^2 - R_0^2} \\ &\geq \frac{\delta h^{\frac{1}{20}-1}}{\mathcal{O}(1)} > 0, \end{aligned}$$

the equation $\widetilde{\beta} = \mathscr{F}^{\delta}(z,\widetilde{\alpha},\alpha')$ has exactly one solution $\widetilde{\alpha}(\widetilde{\beta},\alpha';z)$ in the ball

$$B\left(0, \sqrt{R^2 - \|\alpha'\|^2_{\mathbb{C}^{N-2}}}\right).$$

Moreover, the solution satisfies $\widetilde{\alpha}(\widetilde{\beta},\alpha';z) \in B(0,\sqrt{R^2-R_0^2})$, and it depends holomorphically on $\widetilde{\beta}$ and α' and is smooth in z. Using (3.4.9), we see that the solution is implicitly given by

$$\widetilde{\alpha}(\widetilde{\beta},\alpha') = -\delta^{-1}A^{-\frac{1}{2}}\left(\widetilde{\beta} - v(z,\widetilde{\alpha}(\widetilde{\beta},\alpha'),\alpha',h,\delta)\right). \tag{3.4.17}$$

with

$$v := (v_1, v_2)^t := \mu(z,h) + \tau(z,\widetilde{\alpha}(\widetilde{\beta},\alpha'),\alpha',h,\delta)$$

where τ satisfies the estimate (3.4.10). Since the support of χ is compact (cf. Section 3.1.1), we can restrict our attention to $\widetilde{\beta}$ and $\mathscr{F}^{\delta}(z;0,\alpha')$ in a small poly-disc of radius $K\varepsilon > 0$ centered at 0, with $K > 0$ large enough such that $\operatorname{supp}\chi \subset D(0,K)$. By choosing $\varepsilon < \delta h/C$, $C > 0$ large enough, we see that $\widetilde{\beta}, \mathscr{F}^{\delta}(z;0,\alpha') \in D(0,K\varepsilon) \times D(0,K\varepsilon)$ implies (3.4.16).

From (3.4.8), (3.4.9) and (3.4.17), it follows that

$$\begin{pmatrix} F_3^{\delta}(z,\widetilde{\alpha}(\widetilde{\beta},\alpha'),\alpha') \\ F_4^{\delta}(z,\widetilde{\alpha}(\widetilde{\beta},\alpha'),\alpha') \end{pmatrix} = \partial_z v + B^* A^{-1}(\widetilde{\beta} - v) - \delta\Gamma^{\frac{1}{2}}\begin{pmatrix}\alpha_3 \\ \alpha_4\end{pmatrix}, \tag{3.4.18}$$

with

$$\partial_z v = (\partial_z v_1, \partial_z v_2)^t = (\partial_z\mu)(z,h) + (\partial_z\tau)(z,\widetilde{\alpha}(\widetilde{\beta},\alpha'),\alpha',h,\delta)$$

where $\partial_z\tau$ satisfies the estimate given in (3.4.10). Furthermore, (3.4.13) and (3.4.14) imply that

$$L(d\widetilde{\alpha}) = \delta^{-4}\left(\sqrt{\det A} + \mathcal{O}\left(\delta h^{-\frac{3}{2}}\right)\right)^{-2} L(d\widetilde{\beta}) =: J(\widetilde{\beta},\alpha')L(d\widetilde{\beta}) \tag{3.4.19}$$

By performing this change of variables in the integral I_1 and by picking $\varepsilon > 0$ small enough as above, we get that I_1 is equal to

$$\lim_{\varepsilon\searrow 0}\pi^{-N} \iint\limits_{\substack{\widetilde{\beta}\in D(0,K\varepsilon)\times D(0,K\varepsilon) \\ (\widetilde{\alpha}(\widetilde{\beta},\alpha'),\alpha')\in B(0,R) \\ \|\alpha'\|_{\mathbb{C}^{N-2}}\leq R_0}} H_{\varepsilon}^{\delta}(z_1,z_2,\widetilde{\alpha}(\widetilde{\beta},\alpha'),\alpha';h)\mathrm{e}^{-\Phi(\widetilde{\beta},\alpha')}J(\widetilde{\beta},\alpha')L(d\alpha')L(d\widetilde{\beta}),$$

where

$$\Phi(\widetilde{\beta},\alpha') := \widetilde{\alpha}(\widetilde{\beta},\alpha')^* \cdot \widetilde{\alpha}(\widetilde{\beta},\alpha') + (\alpha')^* \cdot \alpha'.$$

The integrand of I_1 depends continuously on $\widetilde{\beta}$. Hence, by performing the limit $\varepsilon \to 0^+$, we get

$$I_1(z,h) = \pi^{-N} \int\limits_{\substack{(\tilde{\alpha}(0,\alpha'),\alpha')\in B(0,R)\\ \|\alpha'\|_{\mathbb{C}^{N-2}}\leq R_0}} H_0^{\delta}(z_1,z_2,\widetilde{\alpha}(0,\alpha'),\alpha';h)\mathrm{e}^{-\Phi(0,\alpha')}J(0,\alpha')L(d\alpha') \tag{3.4.20}$$

with

$$H_0^{\delta}(z_1,z_2,\widetilde{\alpha}(0,\alpha'),\alpha';h) = |F_3(z,0,\alpha')F_4(z,0,\alpha')|^2.$$

Using (3.4.17), one computes that

$$\Phi(0,\alpha') = \frac{1}{\delta^2}v^* A^{-1} v + (\alpha')^* \cdot \alpha'$$

and, using (3.4.18), we get

$$\begin{pmatrix} F_3^{\delta}(z,\widetilde{\alpha}(0,\alpha'),\alpha') \\ F_4^{\delta}(z,\widetilde{\alpha}(0,\alpha'),\alpha')\end{pmatrix} = \partial_z v - B^* A^{-1} v - \delta\Gamma^{\frac{1}{2}}\begin{pmatrix}\alpha_3\\ \alpha_4\end{pmatrix}, \tag{3.4.21}$$

where $v = v(z,\widetilde{\alpha}(0,\alpha'),\alpha',h,\delta)$. Using (3.4.10), (3.4.11) and (3.4.15) one computes that

$$\|\widetilde{\alpha}(0,\alpha')\|^2 = \frac{1}{\delta^2}v^* A^{-1} v \leq \frac{C}{h^{\frac{1}{10}}}\left[\mathcal{O}\left(\delta^{-2}\mathrm{e}^{-\frac{2S}{h}}\right) + \mathcal{O}\left(\delta^2 h^{-5}\right)\right], \tag{3.4.22}$$

where the constant $C > 0$ comes from the upper bound of $\|A^{-1/2}\|^{-1}$ given in (3.4.15). By the Hypothesis (1.3.2), we conclude that

$$\|\widetilde{\alpha}(0,\alpha')\|^2 \ll \frac{1}{h^{\frac{1}{10}}}.$$

which implies that $(\tilde{\alpha}(0,0,\alpha'),\alpha') \in B(0,R)$ for all α' with $\|\alpha'\|_{\mathbb{C}^{N-2}} \leq R_0$. Hence,

$$I_1(z,h) = \pi^{-N} \int\limits_{\|\alpha'\|_{\mathbb{C}^{N-2}}\leq R_0} |F_3(z,0,\alpha')F_4(z,0,\alpha')|^2 \mathrm{e}^{-\Phi(0,\alpha')}J(0,\alpha')L(d\alpha'). \tag{3.4.23}$$

Next, we want to apply a multi-dimensional version of the mean value theorem for integrals to (3.4.23). Indeed, let $U \subset \mathbb{R}^n$ be open, relatively compact and path-connected, it then holds true that for a continuous function $f : \overline{U} \to \mathbb{R}$ and a positive integrable function $g : \overline{U} \to \mathbb{R}$, there exists a $y \in \overline{U}$ such that

$$f(y)\int_U g(x)dx = \int_U f(x)g(x)dx.$$

Hence, the mean value theorem applied to (3.4.23) yields that

$$I_1(z,h) = \pi^{-N} J \mathrm{e}^{-\frac{\tilde{v}^* A^{-1}\tilde{v}}{\delta^2}} \int\limits_{\|\alpha'\|_{\mathbb{C}^{N-2}}\leq R_0} |F_3(z,0,\alpha')F_4(z,0,\alpha')|^2 \mathrm{e}^{-\alpha'\overline{\alpha'}}L(d\alpha').$$

Here, J denotes the evaluation of the Jacobian $J(0,\alpha')$ (cf. (3.4.19)) at the intermediate point for α' given by mean value theorem. Note that J depends smoothly on z_1 and z_2 because τ and A do.

Similarly, $\widetilde{v}$ above denotes the evaluation of the function $v(z,\widetilde{\alpha}(0,\alpha'),\alpha',h,\delta)$ at the intermediate point for α' given by mean value theorem. It depends smoothly on z_1 and z_2 because μ and τ do. Moreover, using (3.4.10), we see that it satisfies

$$\widetilde{v} = \begin{pmatrix} E_{-+}(z_1)\\ E_{-+}(z_2)\end{pmatrix} + \mathcal{O}\left(\delta^2 h^{-\frac{5}{2}}\right).$$

In remains to study the integral

$$\widetilde{I}_1(z,h) := \pi^{-N} \int\limits_{\|\alpha'\|_{\mathbb{C}^{N-2}}\leq R_0} |F_3(z,0,\alpha')F_4(z,0,\alpha')|^2 \mathrm{e}^{-\alpha'\overline{\alpha'}} L(d\alpha').$$

Define the linear forms

$$l_1(\alpha') = [\Gamma^{\frac{1}{2}}]_{11}\alpha_3 + [\Gamma^{\frac{1}{2}}]_{12}\alpha_4, \quad l_2(\alpha') = [\Gamma^{\frac{1}{2}}]_{21}\alpha_3 + [\Gamma^{\frac{1}{2}}]_{22}\alpha_4.$$

Using (3.4.21), we get that

$$F_3(z,0,\alpha') = (\partial_z v - B^* A^{-1} v)_1 - \delta l_1(\alpha') = \mathcal{O}\left(h^{-\frac{3}{5}}\mathrm{e}^{-\frac{S}{h}} + \delta^2 h^{-\frac{36}{10}}\right) - \delta l_1(\alpha').$$
$$F_4(z,0,\alpha') = (\partial_z v - B^* A^{-1} v)_2 - \delta l_2(\alpha') = \mathcal{O}\left(h^{-\frac{3}{5}}\mathrm{e}^{-\frac{S}{h}} + \delta^2 h^{-\frac{36}{10}}\right) - \delta l_2(\alpha').$$

In the last equation we used (3.4.10), (3.4.11), (3.4.15) and the fact that the Hilbert-Schmidt norm of B^* is $\leq \frac{1}{h\mathcal{O}(1)}$ which follows from the fact that elements of the matrix B^* are bounded by a term of order h^{-1}.

By Proposition 3.3.6, one gets that the Hilbert-Schmidt norm of $\Gamma^{\frac{1}{2}}$ is bounded, indeed one has that

$$\|\Gamma^{\frac{1}{2}}\|_{\mathrm{HS}} = \sqrt{\operatorname{tr}\Gamma} \leq \mathcal{O}(h^{-\frac{1}{2}}).$$

Since $\|\alpha'\|_{\mathbb{C}^{N-2}} \leq R_0$, one gets

$$|F_3(z,0,\alpha')F_4(z,0,\alpha')|^2 = \delta^4\left(|l_1(\alpha')l_2(\alpha')|^2 + \mathcal{O}\left(\mathrm{e}^{-\frac{1}{Ch}} + \delta h^{-\frac{51}{10}}\right)\right),$$

where the error estimate is uniform in α'. Here we used as well that by the hypothesis (1.3.2), we have that $\mathcal{O}(\delta^{-1}\mathrm{e}^{-\frac{S}{h}}) = \mathcal{O}(\mathrm{e}^{-\frac{1}{Ch}})$. Hence,

$$\widetilde{I}_1(z,h) = \delta^4\pi^{-N} \int\limits_{\|\alpha'\|_{\mathbb{C}^{N-2}}\leq R_0} |l_1(\alpha')l_2(\alpha')|^2 \mathrm{e}^{-\alpha'\overline{\alpha'}} L(d\alpha') + \mathcal{O}\left(\delta^4\mathrm{e}^{-\frac{1}{Ch}} + \delta^5 h^{-\frac{51}{10}}\right).$$

Extend the function $|l_1(\alpha')l_2(\alpha')|^2$ to the whole of $\mathbb{C}^{N-2}$ by a function that satisfies the same bounds, i.e. bounded by a term of order h^{-5}, and note that

$$\pi^{2-N} \int\limits_{\|\alpha'\|_{\mathbb{C}^{N-2}}\geq R_0} |l_1(\alpha')l_2(\alpha')|^2 \mathrm{e}^{-\alpha'\overline{\alpha'}} L(d\alpha') \leq \mathcal{O}\left(\mathrm{e}^{-\frac{D}{h^2}}\right).$$

Integration by parts yields that

$$\pi^{2-N} \int\limits_{\mathbb{C}^{N-2}} |l_1(\alpha')l_2(\alpha')|^2 \mathrm{e}^{-\alpha'\overline{\alpha'}} L(d\alpha')$$
$$= \pi^{-2} \int_{\mathbb{C}^{N-2}} \mathrm{e}^{-\widetilde{\alpha}\overline{\widetilde{\alpha}}} \prod_{k=1}^{2} l_k(\overline{\partial_{\widetilde{\alpha}}}) \left(\prod_{n=1}^{2} \overline{l_n}(\overline{\widetilde{\alpha}})\right) L(d\widetilde{\alpha}).$$

Note that for any permutation $\sigma \in S_n$, where S_n is the symmetric group, we have that $(l_i|l_{\sigma(i)}) = \Gamma_{i\sigma(i)}$. Thus, in view of (3.1.8), we have that

$$\prod_{k=1}^{2} l_k(\overline{\partial_{\widetilde{\alpha}}}) \left(\prod_{n=1}^{2} \overline{l_n}(\overline{\widetilde{\alpha}})\right) = \sum_{\sigma\in S_2} (l_1|l_{\sigma(1)})(l_2|l_{\sigma(2)}) = \operatorname{perm}\Gamma$$

We conclude that

$$I_1(z,h) = \frac{\operatorname{perm}\Gamma + \mathcal{O}\left(\mathrm{e}^{-\frac{1}{Ch}} + \delta h^{-\frac{51}{10}}\right)}{\pi^2\left(\sqrt{\det A} + \mathcal{O}\left(\delta h^{-\frac{3}{2}}\right)\right)^2} + \mathcal{O}\left(\mathrm{e}^{-\frac{D}{h^2}}\right),$$

where we used the fact that $\det A \geq \frac{h^{\frac{1}{5}}}{\mathcal{O}(1)}$ for $1/C \geq |z-w| \gg h^{3/5}$, see Proposition 3.3.4, to obtain the last equality.

The integral I_2 In this step we will estimate the second integral of equation (3.4.12). Therefore, we will increase the space of integration

$$\pi^{-N} \int_{\substack{B(0,R) \\ R_0 < \|\alpha'\|_{\mathbb{C}^{N-2}} < R}} \prod_{k=1}^{2} \varepsilon^{-2} \chi\left(\frac{F_k(z,\alpha)}{\varepsilon}\right) |\partial_{z_k} F_k(z,\alpha)|^2 \mathrm{e}^{-\alpha\overline{\alpha}} L(d\alpha)$$
$$\leq \pi^{-N} \int_{\substack{B(0,2R) \\ R_0 < \|\alpha'\|_{\mathbb{C}^{N-2}} < 2R_0}} \prod_{k=1}^{2} \varepsilon^{-2} \chi\left(\frac{F_k(z,\alpha)}{\varepsilon}\right) |\partial_{z_k} F_k(z,\alpha)|^2 \mathrm{e}^{-\alpha\overline{\alpha}} L(d\alpha) =: W_\varepsilon.$$

It is easy to see that Lemma 3.4.1 holds true for the set $B(0,2R) \cap \{R_0 < \|\alpha'\|_{\mathbb{C}^{N-2}} < 2R_0\}$. Therefore, we can proceed as for the integral I_1: perform the same change of variables and perform the limit of $\varepsilon \to 0$. As for I_1, the integrand remains bounded by at most a finite power of h^{-1} which then yields that

$$\lim_{\varepsilon\to 0} W_\varepsilon = \mathcal{O}\left(\mathrm{e}^{-\frac{D}{h^2}}\right),$$

where the exponential decay comes from the fact that $R_0 < \|\alpha'\|_{\mathbb{C}^{N-2}}$. Therefore,

$$\int_{\mathbb{C}^2} \varphi_1(z_1)\varphi_2(z_2) d\nu(z_1,z_2) = \int_{\mathbb{C}^2} \varphi_1(z_1)\varphi_2(z_2) D(z,h) L(dz_1 dz_2)$$

with

$$D(z,h,\delta) = \frac{\operatorname{perm}\Gamma + \mathcal{O}\left(\mathrm{e}^{-\frac{1}{Ch}} + \delta h^{-\frac{51}{10}}\right)}{\pi^2\left(\sqrt{\det A} + \mathcal{O}\left(\delta h^{-\frac{3}{2}}\right)\right)^2} + \mathcal{O}\left(\mathrm{e}^{-\frac{D}{h^2}}\right). \qquad \square$$

CHAPTER 4

INTERIOR EIGENVALUE DENSITY OF JORDAN MATRICES WITH RANDOM PERTURBATIONS

The aim of this chapter is to study the eigenvalue distribution of a large Jordan block subject to a small random Gaussian perturbation, as was discussed in Section 1.4, and give a precise asymptotic description of the expected eigenvalue density in the interior of a circle thereby extending an existing result of E.B. Davies and M. Hager [16]. In particular, we prove the results described in Section 1.4. The results presented here are due to J. Sjöstrand and M. Vogel [71].

4.1 | A general formula

To start with, we shall obtain a general formula (due to [83] in a similar context). Our treatment is slightly different in that we avoid the use of approximations of the delta function and also that we have more holomorphy available.

Let $g(z,Q)$ be a holomorphic function on $\Omega\times W\subset\mathbb{C}\times\mathbb{C}^{N^2}$, where $\Omega\subset\mathbb{C}$, $W\subset\mathbb{C}^{N^2}$ are open bounded and connected. Assume that

$$\text{for every } Q\in W,\ g(\cdot,Q)\not\equiv 0. \tag{4.1.1}$$

To start with, we also assume that

$$\text{for almost all } Q\in W,\ g(\cdot,Q) \text{ has only simple zeros.} \tag{4.1.2}$$

Let $\phi\in C_0^\infty(\Omega)$ and let $m\in C_0(W)$. We are interested in

$$K_\phi=\int\left(\sum_{z;\,g(z,Q)=0}\phi(z)\right)m(Q)L(dQ), \tag{4.1.3}$$

where we frequently identify the Lebesgue measure with a differential form,

$$L(dQ)\simeq(2i)^{-N^2}d\overline{Q}_1\wedge dQ_1\wedge\ldots\wedge d\overline{Q}_{N^2}\wedge dQ_{N^2}=:(2i)^{-N^2}d\overline{Q}\wedge dQ.$$

In (4.1.3) we count the zeros of $g(\cdot,Q)$ with their multiplicity and notice that the integral is finite: For every compact set $K\subset W$ the number of zeros of $g(\cdot,Q)$ in $\operatorname{supp}\phi$, counted with their multiplicity, is uniformly bounded, for $Q\in K$. This follows from Jensen's formula.

Now assume,

$$g(z,Q)=0\Rightarrow d_Q g\neq 0. \tag{4.1.4}$$

Then

$$\Sigma := \{(z,Q) \in \Omega \times W;\, g(z,Q) = 0\}$$

is a smooth complex hypersurface in $\Omega \times W$ and from (4.1.2) we see that

$$K_\phi = \int_\Sigma \phi(z) m(Q) (2i)^{-N^2} d\overline{Q} \wedge dQ, \tag{4.1.5}$$

where we view $(2i)^{-N^2} d\overline{Q} \wedge dQ$ as a complex (N^2, N^2)-form on $\Omega \times W$, restricted to Σ, which yields a non-negative differential form of maximal degree on Σ.

Before continuing, let us eliminate the assumption (4.1.2). Without that assumption, the integral in (4.1.3) is still well-defined. It suffices to show (4.1.5) for all $\phi \in \mathscr{C}_0^\infty(\Omega_0 \times W_0)$ when $\Omega_0 \times W_0$ is a sufficiently small open neighborhood of any given point $(z_0, Q_0) \in \Omega \times W$. When $g(z_0, Q_0) \neq 0$ or $\partial_z g(z_0, \Omega_0) \neq 0$ we already know that this holds, so we assume that for some $m \geq 2$, $\partial_z^k g(z_0, Q_0) = 0$ for $0 \leq k \leq m-1$, $\partial_z^m g(z_0, Q_0) \neq 0$.

Put $g_\varepsilon(z,Q) = g(z,Q) + \varepsilon$, $\varepsilon \in \mathrm{neigh}(0, \mathbb{C})$. By Weierstrass' preparation theorem, if Ω_0, W_0 and $r > 0$ are small enough,

$$g_\varepsilon(z,Q) = k(z,Q,\varepsilon) p(z,Q,\varepsilon) \quad \text{in } \Omega_0 \times W_0 \times D(0,r),$$

where k is holomorphic and non-vanishing, and

$$p(z,Q,\varepsilon) = z^m + p_1(Q,\varepsilon) z^{m-1} + \cdots + p_m(Q,\varepsilon).$$

Here, $p_j(Q,\varepsilon)$ are holomorphic, and $p_j(0,0) = 0$. The discriminant $D(Q,\varepsilon)$ of the polynomial $p(\cdot,Q,\varepsilon)$ is holomorphic on $W_0 \times D(0,r)$. It vanishes precisely when $p(\cdot,Q,\varepsilon)$ - or equivalently $g_\varepsilon(\cdot,Q)$ - has a multiple root in Ω_0.

Now for $0 < |\varepsilon| \ll 1$, the m roots of $g_\varepsilon(\cdot,Q_0)$ are simple, so $D(Q_0,\varepsilon) \neq 0$. Thus, $D(\cdot,\varepsilon)$ is not identically zero, so the zero set of $D(\cdot,\varepsilon)$ in W_0 is of measure 0 (assuming that we have chosen W_0 connected). This means that for $0 < |\varepsilon| \ll 1$, the function $g_\varepsilon(\cdot,Q)$ has only simple roots in Ω for almost all $Q \in W_0$.

Let Σ_ϵ be the zero set of g_ϵ, so that $\Sigma_\epsilon \to \Sigma_0 = \Sigma \cap (\Omega_0 \times W_0)$ uniformly. We have

$$\int \left(\sum_{z;\, g_\epsilon(z,Q)=0} \phi(z) \right) m(Q)(2i)^{-N^2} d\overline{Q} \wedge dQ = \int_{\Sigma_\epsilon} \phi(z) m(Q)(2i)^{-N^2} d\overline{Q} \wedge dQ$$

for $\phi \in \mathscr{C}_0^\infty(\Omega_0 \times W_0)$, when $\epsilon > 0$ is small enough, depending on ϕ, m. Passing to the limit $\epsilon = 0$ we get (4.1.5) under the assumptions (4.1.1), (4.1.4), first for $\phi \in \mathscr{C}_0^\infty(\Omega_0 \times W_0)$, and then by partition of unity for all $\phi \in \mathscr{C}_0^\infty(\Omega \times W)$. Notice that the result remains valid if we replace $m(Q)$ by $m(Q) 1_B(Q)$ where B is a ball in W.

Now we strengthen the assumption (4.1.4) by assuming that we have a non-zero $Z(z) \in \mathbb{C}^{N^2}$ depending smoothly on $z \in \Omega$ (the dependence will actually be holomorphic in the application below) such that

$$g(z,Q) = 0 \Rightarrow \left(\overline{Z}(z) \cdot \partial_Q \right) g(z,Q) \neq 0. \tag{4.1.6}$$

We have the corresponding orthogonal decomposition

$$Q = Q(\alpha) = \alpha_1 \overline{Z}(z) + \alpha', \quad \alpha' \in \overline{Z}(z)^\perp,\ \alpha_1 \in \mathbb{C},$$

and if we identify unitarily $\overline{Z}(z)^\perp$ with $\mathbb{C}^{N^2-1}$ by means of an orthonormal basis

$$e_2(z), \ldots, e_{N^2}(z),$$

so that $\alpha' = \sum_2^{N^2} \alpha_j e_j(z)$ we get global coordinates $\alpha_1, \alpha_2, \ldots, \alpha_{N^2}$ on Q-space (i.e. W).

By the implicit function theorem, at least locally near any given point in Σ, we can represent Σ by $\alpha_1 = f(z,\alpha')$, $\alpha' \in \overline{Z}(z)^\perp \simeq \mathbb{C}^{N^2-1}$, where f is smooth. (In the specific situation below, this will be valid globally.) Clearly, since $z, \alpha_2, \ldots, \alpha_{N^2}$ are complex coordinates on Σ, we have on Σ that

$$\frac{1}{(2i)^{N^2}} d\overline{Q} \wedge dQ = J(f) \frac{d\overline{z} \wedge dz}{2i} (2i)^{1-N^2} d\overline{\alpha}_2 \wedge d\alpha_2 \wedge \ldots \wedge d\overline{\alpha}_{N^2} \wedge d\alpha_{N^2},$$

where we view $(2i)^{-N^2} d\overline{Q}\wedge dQ$ as a complex (N^2,N^2)-form on $\Omega\times W$, restricted to Σ (as in (4.1.5)), and we use the convention that

$$J(f)\frac{d\overline{z}\wedge dz}{2i}\geq 0,\quad (2i)^{1-N^2} d\overline{\alpha}_2\wedge d\alpha_2\wedge\ldots\wedge d\overline{\alpha}_{N^2}\wedge d\alpha_{N^2}>0.$$

Thus

$$\begin{aligned}K_\phi=\int\phi(z)m\left(f(z,\alpha')\overline{Z}(z)+\alpha'\right)J(f)(z,\alpha_2,\ldots,\alpha_{N^2})\times\\ (2i)^{-N^2} d\overline{z}\wedge dz\wedge d\overline{\alpha}_2\wedge d\alpha_2\wedge\ldots\wedge d\overline{\alpha}_{N^2}\wedge d\alpha_{N^2}.\end{aligned}\tag{4.1.7}$$

The Jacobian $J(f)$ is invariant under any z-dependent unitary change of variables, $\alpha_2,\ldots,\alpha_{N^2}\mapsto\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_{N^2}$, so for the calculation of $J(f)$ at a given point (z_0,α_0'), we are free to choose the most appropriate orthonormal basis $e_2(z),\ldots,e_{N^2}(z)$ in $\overline{Z}(z)^\perp$ depending smoothly on z. We write (4.1.7) as

$$K_\phi=\int\phi(z)\widetilde{\Xi}(z)\frac{d\overline{z}\wedge dz}{2i},\tag{4.1.8}$$

where the density $\widetilde{\Xi}(z)$ is given by

$$\begin{aligned}\widetilde{\Xi}(z)=\int_{\alpha'=\sum_2^{N^2}\alpha_j e_j(z)} m(f(z,\alpha')\overline{Z}(z)+\alpha')J(f)(z,\alpha_2,\ldots,\alpha_{N^2})\times\\ (2i)^{1-N^2} d\overline{\alpha}_2\wedge d\alpha_2\wedge\ldots\wedge d\overline{\alpha}_{N^2}\wedge d\alpha_{N^2}.\end{aligned}\tag{4.1.9}$$

4.2 | Grushin problem for the perturbed Jordan block

4.2.1 – Setting up an auxiliary problem

Following [74], we introduce an auxiliary Grushin problem. Define $R_+:\mathbb{C}^N\to\mathbb{C}$ by

$$R_+u=u_1,\ u=(u_1\ \ldots\ u_N)^{\mathrm{t}}\in\mathbb{C}^N.\tag{4.2.1}$$

Let $R_-:\mathbb{C}\to\mathbb{C}^N$ be defined by

$$R_-u_-=(0\ 0\ \ldots\ u_-)^{\mathrm{t}}\in\mathbb{C}^N.\tag{4.2.2}$$

Here, we identify vectors in $\mathbb{C}^N$ with column matrices. Then for $|z|<1$, the operator

$$\mathscr{A}_0=\begin{pmatrix}A_0-z & R_-\\ R_+ & 0\end{pmatrix}:\mathbb{C}^{N+1}\to\mathbb{C}^{N+1}\tag{4.2.3}$$

is bijective. In fact, identifying

$$\mathbb{C}^{N+1}\simeq\ell^2([1,2,\ldots,N+1])\simeq\ell^2(\mathbb{Z}/(N+1)\mathbb{Z}),$$

we have $\mathscr{A}_0=\tau^{-1}-z\Pi_N$, where $\tau u(j)=u(j-1)$ (translation by 1 step to the right, keeping in mind that $j\in\mathbb{Z}/(N+1)\mathbb{Z}$) and $\Pi_N u=1_{[1,N]}u$. Then $\mathscr{A}_0=\tau^{-1}(1-z\tau\Pi_N)$, $(\tau\Pi_N)^{N+1}=0$,

$$\mathscr{A}_0^{-1}=(1+z\tau\Pi_N+(z\tau\Pi_N)^2+\ldots+(z\tau\Pi_N)^N)\circ\tau.$$

Write

$$\mathscr{E}_0:=\mathscr{A}_0^{-1}=:\begin{pmatrix}E^0 & E^0_+\\ E^0_- & E^0_{-+}\end{pmatrix}.$$

Then

$$E^0\simeq\Pi_N(1+z\tau\Pi_N+\ldots(z\tau\Pi_N)^{N-1})\tau\Pi_N,\tag{4.2.4}$$

$$E_+^0 = \begin{pmatrix} 1 \\ z \\ .. \\ z^{N-1} \end{pmatrix}, \quad E_-^0 = \begin{pmatrix} z^{N-1} & z^{N-2} & \dots & 1 \end{pmatrix}, \tag{4.2.5}$$

$$E_{-+}^0 = z^N. \tag{4.2.6}$$

A quick way to check (4.2.5), (4.2.6) is to write $\mathscr{A}_0$ as an $(N+1)\times(N+1)$-matrix where we moved the last line to the top, with the lines labeled from 0 ($\equiv N+1 \mod (N+1)\mathbb{Z}$) to N and the columns from 1 to $N+1$.

Continuing, we see that

$$\|E^0\| \le G(|z|),\ \|E_\pm^0\| \le G(|z|)^{\frac{1}{2}},\ \|E_{-+}^0\| \le 1, \tag{4.2.7}$$

where $\|\cdot\|$ denote the natural operator norms and

$$G(|z|) := \min\left(N, \frac{1}{1-|z|}\right) \asymp 1 + |z| + |z|^2 + \dots + |z|^{N-1}. \tag{4.2.8}$$

Next, consider the natural Grushin problem for A_δ. If $\delta\|Q\|G(|z|) < 1$, we see that

$$\mathscr{A}_\delta = \begin{pmatrix} A_\delta - z & R_- \\ R_+ & 0 \end{pmatrix} \tag{4.2.9}$$

is bijective with inverse

$$\mathscr{E}_\delta = \begin{pmatrix} E^\delta & E_+^\delta \\ E_-^\delta & E_{-+}^\delta \end{pmatrix},$$

where

$$\begin{aligned} E^\delta &= E^0 - E^0\delta Q E^0 + E^0(\delta Q E^0)^2 - \dots = E^0(1+\delta Q E^0)^{-1}, \\ E_+^\delta &= E_+^0 - E^0\delta Q E_+^0 + (E^0\delta Q)^2 E_+^0 - \dots = (1+E^0\delta Q)^{-1}E_+^0, \\ E_-^\delta &= E_-^0 - E_-^0\delta Q E^0 + E_-^0(\delta Q E^0)^2 - \dots = E_-^0(1+\delta Q E^0)^{-1}, \\ E_{-+}^\delta &= E_{-+}^0 - E_-^0\delta Q E_+^0 + E_-^0\delta Q E^0 \delta Q E_+^0 - \dots \\ &= E_{-+}^0 - E_-^0\delta Q(1+E^0\delta Q)^{-1}E_+^0. \end{aligned} \tag{4.2.10}$$

We get

$$\begin{aligned} &\|E^\delta\| \le \frac{G(|z|)}{1-\delta\|Q\|G(|z|)},\ \|E_\pm^\delta\| \le \frac{G(|z|)^{\frac{1}{2}}}{1-\delta\|Q\|G(|z|)}, \\ &|E_{-+}^\delta - E_{-+}^0| \le \frac{\delta\|Q\|G(|z|)}{1-\delta\|Q\|G(|z|)}. \end{aligned} \tag{4.2.11}$$

Indicating derivatives with respect to δ with dots and omitting sometimes the super- or sub-script δ, we have

$$\dot{\mathscr{E}} = -\mathscr{E}\dot{\mathscr{A}}\mathscr{E} = -\begin{pmatrix} EQE & EQE_+ \\ E_-QE & E_-QE_+. \end{pmatrix} \tag{4.2.12}$$

Integrating this from 0 to δ yields

$$\|E^\delta - E^0\| \le \frac{G(|z|)^2\delta\|Q\|}{(1-\delta\|Q\|G(|z|))^2}, \quad \|E_\pm^\delta - E_\pm^0\| \le \frac{G(|z|)^{\frac{3}{2}}\delta\|Q\|}{(1-\delta\|Q\|G(|z|))^2}. \tag{4.2.13}$$

We now sharpen the assumption that $\delta\|Q\|G(|z|) < 1$ to

$$\delta\|Q\|G(|z|) < 1/2. \tag{4.2.14}$$

Then

$$\begin{aligned} &\|E^\delta\| \le 2G(|z|),\ \|E^\delta_\pm\| \le 2G(|z|)^{\frac{1}{2}},\\ &|E^\delta_{-+} - E^0_{-+}| \le 2\delta\|Q\|G(|z|). \end{aligned} \tag{4.2.15}$$

Combining this with the identity $\dot{E}_{-+} = -E_- Q E_+$ (recall that here the dot indicates a derivative with respect to δ) that follows from (4.2.12), we get

$$\|\dot{E}_{-+} + E^0_- Q E^0_+\| \le 16G(|z|)^2\delta\|Q\|^2, \tag{4.2.16}$$

and after integration from 0 to δ,

$$E^\delta_{-+} = E^0_{-+} - \delta E^0_- Q E^0_+ + \mathcal{O}(1)G(|z|)^2(\delta\|Q\|)^2. \tag{4.2.17}$$

Using (4.2.5), (4.2.6) we get with $Q = (q_{j,k})$,

$$E^\delta_{-+} = z^N - \delta \sum_{j,k=1}^{N} q_{j,k} z^{N-j+k-1} + \mathcal{O}(1)G(|z|)^2(\delta\|Q\|)^2, \tag{4.2.18}$$

still under the assumption (4.2.14).

4.2.2 – Estimates for the effective Hamiltonian

We now consider the situation of (1.4.2):

$$A_\delta = A_0 + \delta Q,\ Q = (q_{j,k}(\omega))^N_{j,k=1},\ q_{j,k}(\omega) \sim \mathcal{N}_{\mathbb{C}}(0,1) \text{ independent.}$$

W. Bordeaux-Montrieux [4] obtained the following result.

Proposition 4.2.1. *There exists a $C_0 > 0$ such that the following holds: Let*

$$X_j \sim \mathcal{N}_{\mathbb{C}}(0,\sigma_j^2), \quad 1 \le j \le N < \infty$$

be independent complex Gaussian random variables. Put $s_1 = \max \sigma_j^2$. Then, for every $x > 0$, we have

$$\mathbb{P}\left[\sum_{j=1}^N |X_j|^2 \ge x\right] \le \exp\left(\frac{C_0}{2s_1}\sum_{j=1}^N \sigma_j^2 - \frac{x}{2s_1}\right).$$

According to this result we have

$$P(\|Q\|^2_{\mathrm{HS}} \ge x) \le \exp\left(\frac{C_0}{2}N^2 - \frac{x}{2}\right)$$

and hence if $C_1 > 0$ is large enough,

$$\|Q\|^2_{\mathrm{HS}} \le C_1^2 N^2, \text{ with probability } \ge 1 - e^{-N^2}. \tag{4.2.19}$$

In particular (4.2.19) holds for the ordinary operator norm of Q. In the following, we often write $|\cdot|$ for the Hilbert-Schmidt norm $\|\cdot\|_{\mathrm{HS}}$ and we shall work under the assumption that $|Q| \le C_1 N$. We let $|z| < 1$ and assume:

$$\delta N G(|z|) \ll 1. \tag{4.2.20}$$

Then with probability $\ge 1 - e^{-N^2}$, we have (4.2.14), (4.2.18) which give for $g(z,Q) := E^\delta_{-+}$,

$$g(z,Q) = z^N - \delta(Q|\overline{Z}(z)) + \mathcal{O}(1)(G(|z|)\delta N)^2. \tag{4.2.21}$$

Here, Z is given by

$$Z = \left(z^{N-j+k-1}\right)^N_{j,k=1}. \tag{4.2.22}$$

Remark 4.2.2. The above Z will play in the following the role of the Z in (4.1.6).

A straight forward calculation shows that

$$|Z| = \sum_0^{N-1} |z|^{2\nu} = \frac{1-|z|^{2N}}{1-|z|^2} = \frac{1-|z|^N}{1-|z|}\frac{1+|z|^N}{1+|z|}, \tag{4.2.23}$$

and in particular,

$$\frac{G(|z|)}{2} \le |Z| \le G(|z|). \tag{4.2.24}$$

The middle term in (4.2.21) is bounded in modulus by $\delta|Q||Z| \le \delta C_1 N G(|z|)$ and we assume that $|z|^N$ is much smaller than this bound:

$$|z|^N \ll \delta C_1 N G(|z|). \tag{4.2.25}$$

More precisely, we work in a disc $D(0, r_0)$, where

$$r_0^N \le C^{-1}\delta C_1 N G(r_0) \le C^{-2}, \quad r_0 \le 1 - N^{-1} \tag{4.2.26}$$

and $C \gg 1$. In fact, the first inequality in (4.2.26) can be written $m(r_0) \le C^{-1}\delta C_1 N$ and $m(r) = r^N(1-r)$ is increasing on $[0, 1-N^{-1}]$ so the inequality is preserved if we replace r_0 by $|z|$ for $|z| \le r_0$. Similarly, the second inequality holds after the same replacement since G is increasing.

In view of (4.2.20), we see that

$$(G(|z|)\delta N)^2 \ll \delta G(|z|) N$$

is also much smaller than the upper bound on the middle term.

By the Cauchy inequalities,

$$d_Q g = -\delta Z \cdot dQ + \mathcal{O}(1) G(|z|)^2 \delta^2 N. \tag{4.2.27}$$

The norm of the first term is $\asymp \delta G \gg G^2\delta^2 N$, since $G\delta N \ll 1$. (When applying the Cauchy inequalities, we should shrink the radius $R = C_1 N$ by a factor $\theta < 1$, but we have room for that, if we let C_1 be a little larger than necessary to start with.)

Writing

$$Q = \alpha_1 \overline{Z}(z) + \alpha', \ \alpha' \in \overline{Z}(z)^{\perp} \simeq \mathbb{C}^{N^2-1},$$

we identify $g(z, Q)$ with a function $\widetilde{g}(z, \alpha)$ which is holomorphic in α for every fixed z and satisfies

$$\widetilde{g}(z, \alpha) = z^N - \delta|Z(z)|^2 \alpha_1 + \mathcal{O}(1) G(|z|)^2 \delta^2 N^2, \tag{4.2.28}$$

while (4.2.27) gives

$$\partial_{\alpha_1} \widetilde{g}(z, \alpha) = -\delta |Z(z)|^2 + \mathcal{O}(1) G(|z|)^3 \delta^2 N, \tag{4.2.29}$$

and in particular,

$$\left|\partial_{\alpha_1} \widetilde{g}\right| \asymp \delta G(|z|)^2.$$

This derivative does not depend on the choice of unitary identification $\overline{Z}^{\perp} \simeq \mathbb{C}^{N^2-1}$. Notice that the remainder in (4.2.28) is the same as in (4.2.21) and hence a holomorphic function of (z, Q). In particular it is a holomorphic function of $\alpha_1, \dots, \alpha_{N^2}$ for every fixed z and we can also get (4.2.29) from this and the Cauchy inequalities. In the same way, we get from (4.2.28) that

$$\partial_{\alpha_j} \widetilde{g}(z, \alpha) = \mathcal{O}(1) G(|z|)^2 \delta^2 N, \ j = 2, \dots, N^2. \tag{4.2.30}$$

The Cauchy inequalities applied to (4.2.21) give,

$$\partial_z g(z, Q) = N z^{N-1} - \delta Q \cdot \partial_z Z(z) + \mathcal{O}(1)\frac{(G(|z|)\delta N)^2}{r_0 - |z|}. \tag{4.2.31}$$

Lemma 4.2.3. *For* $\widetilde{g}(z,\alpha_1,\alpha') = g(z,\alpha_1\overline{Z}(z)+\alpha')$, $\alpha' = \sum_2^{N^2}\alpha_j e_j$ *we have that*

$$\partial_z\widetilde{g} = Nz^{N-1} - \delta\alpha_1\partial_z\left(|Z|^2\right) + \mathcal{O}(1)\frac{(G\delta N)^2}{r_0-|z|} + \mathcal{O}(1)G^2\delta^2 N\left|\sum_2^{N^2}\alpha_j\partial_z e_j\right|, \tag{4.2.32}$$

$$\partial_{\overline{z}}\widetilde{g} = -\delta\alpha_1\partial_{\overline{z}}\left(|Z|^2\right) + \mathcal{O}(1)G^2\delta^2 N\left|\alpha_1\overline{\partial_z Z} + \sum_2^{N^2}\alpha_j\partial_{\overline{z}}e_j\right|. \tag{4.2.33}$$

Proof. The leading terms in (4.2.32), (4.2.33) can be obtained formally from (4.2.28) by applying ∂_z, $\partial_{\overline{z}}$ and we also notice that

$$\partial_z|Z|^2 = \overline{Z}\cdot\partial_z Z,\ \ \partial_{\overline{z}}|Z|^2 = Z\cdot\overline{\partial_z Z}.$$

However it is not clear how to handle the remainder in (4.2.28), so we verify (4.2.32), (4.2.33), using (4.2.27), (4.2.31):

$$\begin{aligned}
\partial_z\widetilde{g} &= \partial_z g + d_Q g\cdot\sum_2^{N^2}\alpha_j\partial_z e_j \\
&= Nz^{N-1} - \delta Q\cdot\partial_z Z + \mathcal{O}(1)\frac{(G\delta N)^2}{r_0-|z|} + (-\delta Z\cdot dQ + \mathcal{O}(1)G^2\delta^2 N)\cdot\sum_2^{N^2}\alpha_j\partial_z e_j \\
&= Nz^{N-1} - \delta\alpha_1\partial_z\left(|Z|^2\right) - \delta\sum_2^{N^2}\alpha_j e_j\cdot\partial_z Z - \delta Z\cdot\sum_2^{N^2}\alpha_j\partial_z e_j \\
&\qquad + \mathcal{O}(1)\frac{(G\delta N)^2}{r_0-|z|} + \mathcal{O}(1)G^2\delta^2 N\left|\sum_2^{N^2}\alpha_j\partial_z e_j\right|.
\end{aligned}$$

The 3d and the 4th terms on the right hand side of the last expression add up to

$$\delta\partial_z\left(\sum_2^{N^2}\alpha_j e_j\cdot Z\right) = \delta\partial_z(0) = 0,$$

and we get (4.2.32).

Similarly,

$$\begin{aligned}
\partial_{\overline{z}}\widetilde{g} &= d_Q g\cdot\left(\alpha_1\overline{\partial_z Z} + \sum_2^{N^2}\alpha_j\partial_{\overline{z}}e_j\right) \\
&= \left(-\delta Z\cdot dQ + \mathcal{O}(1)G^2\delta^2 N\right)\cdot\left(\alpha_1\overline{\partial_z Z} + \sum_2^{N^2}\alpha_j\partial_{\overline{z}}e_j\right).
\end{aligned}$$

Up to remainders as in (4.2.33), this is equal to

$$\begin{aligned}
-\delta\alpha_1 Z\cdot\overline{\partial_z Z} - \delta\sum_2^{N^2}\alpha_j Z\cdot\partial_{\overline{z}}e_j &= -\delta\alpha_1\partial_{\overline{z}}\left(|Z|^2\right) - \delta\sum_2^{N^2}\alpha_j\partial_{\overline{z}}\left(Z\cdot e_j\right) \\
&= -\delta\alpha_1\partial_{\overline{z}}\left(|Z|^2\right).
\end{aligned}$$

□

Continuing, we know that

$$|Z(z)| = \sum_0^{N-1}(z\overline{z})^\nu =: K(z\overline{z}), \tag{4.2.34}$$

$$\begin{aligned}
\partial_z\left(|Z(z)|^2\right) &= 2KK'\overline{z}, \\
\partial_{\overline{z}}\left(|Z(z)|^2\right) &= 2KK'z.
\end{aligned} \tag{4.2.35}$$

Observe also that $K(t) \asymp G(t)$ and that $G(|z|) \asymp G(|z|^2)$.

The following result implies that $K'(t)$ and $K(t)^2$ are of the same order of magnitude.

Proposition 4.2.4. *Let K be as in (4.2.34). For $k \in \mathbb{N}$, $2 \le N \in \mathbb{N} \cup \{+\infty\}$, $0 \le t < 1$, we put*

$$M_{N,k}(t) = \sum_{\nu=1}^{N-1} \nu^k t^\nu, \tag{4.2.36}$$

so that

$$K(t) = K_N(t) = M_{N,0}(t) + 1, \quad K'(t) \asymp M_{N-1,1}(t) + 1.$$

For each fixed $k \in \mathbb{N}$, we have uniformly with respect to N, t:

$$M_{\infty,k}(t) \asymp \frac{t}{(1-t)^{k+1}}, \tag{4.2.37}$$

$$M_{\infty,k}(t) - M_{N,k}(t) \asymp \frac{t^N}{1-t}\left(N + \frac{1}{1-t}\right)^k. \tag{4.2.38}$$

For all fixed $C > 0$ and $k \in \mathbb{N}$, we have uniformly,

$$M_{N,k}(t) \asymp M_{\infty,k}(t), \ \textit{for } 0 \le t \le 1 - \frac{1}{CN}, \ N \ge 2. \tag{4.2.39}$$

Notice that under the assumption on t in (4.2.39), the estimate (4.2.38) becomes

$$M_{\infty,k}(t) - M_{N,k}(t) \asymp \frac{t^N N^k}{1-t}.$$

We also see that in any region $1 - \mathcal{O}(1)/N \le t < 1$, we have

$$M_{N,k}(t) \asymp N^{k+1},$$

so together with (4.2.39), (4.2.37), this shows that

$$M_{N,k}(t) \asymp t \min\left(\frac{1}{1-t}, N\right)^{k+1}. \tag{4.2.40}$$

Proof. The statements are easy to verify when $0 \le t \le 1 - 1/\mathcal{O}(1)$ and the N-dependent statements (4.2.38), (4.2.39) are clearly true when $N \le \mathcal{O}(1)$. Thus we can assume that $1/2 \le t < 1$ and $N \gg 1$.

Write $t = e^{-s}$ so that $0 < s \le 1/\mathcal{O}(1)$ and notice that $s \asymp 1 - t$. For $N \in \mathbb{N}$, we put

$$P_{N,k}(s) = \sum_{\nu=N}^{\infty} \nu^k e^{-\nu s}, \tag{4.2.41}$$

so that

$$P_{N,k}(s) = \begin{cases} M_{\infty,k}(t) \text{ when } N = 1, \\ M_{\infty,k}(t) - M_{N,k}(t) \text{ when } N \ge 2. \end{cases} \tag{4.2.42}$$

We regroup the terms in (4.2.41) into sums with $\asymp 1/s$ terms where $e^{-\nu s}$ has constant order of magnitude:

$$P_{N,k}(s) = \sum_{\mu=1}^{\infty} \Sigma(\mu), \quad \Sigma(\mu) = \sum_{N+\frac{\mu-1}{s} \le \nu < N+\frac{\mu}{s}} \nu^k e^{-\nu s}.$$

Here, since the sum $\Sigma(\mu)$ consists of $\asymp 1/s$ terms of the order $\nu^k e^{-(Ns+\mu)}$,

$$\Sigma(\mu) \asymp e^{-(Ns+\mu)} \sum_{N+\frac{\mu-1}{s} \le \nu < N+\frac{\mu}{s}} \nu^k \asymp e^{-(Ns+\mu)} \frac{(Ns+\mu)^k}{s^{k+1}}.$$

Hence,

$$\begin{aligned} P_{N,k}(s) &\asymp \frac{e^{-Ns}}{s^{k+1}} \sum_{\mu=1}^{\infty} e^{-\mu}(Ns+\mu)^k \\ &\asymp \frac{e^{-Ns}}{s^{k+1}}(Ns+1)^k = \frac{e^{-Ns}}{s}\left(N + \frac{1}{s}\right)^k. \end{aligned}$$

Recalling (4.2.42) and the fact that $s \asymp 1-t$, $1/2 \le t < 1$, we get (4.2.37) when $N=1$ and (4.2.38) when $N \ge 2$.

It remains to show (4.2.39) and it suffices to do so for $1/2 \le t \le 1-C/N$, $N \gg 1$ and for $C \ge 1$ sufficiently large but independent of N. Indeed, for $1-C/N \le t \le 1-1/\mathcal{O}(N)$, both $M_{N,k}(t)$ and $M_{\infty,k}(t)$ are $\asymp N^{1+k}$. We can also exclude the case $k=0$ where we have explicit formulae.

To get the equivalence (4.2.39) for $1/2 \le t \le 1-C/N$, $k \ge 1$, it suffices, in view of (4.2.37), (4.2.38), to show that for such t and for $N \gg 1$, we have

$$\frac{N^k t^N}{1-t} \le \frac{1}{D}\frac{1}{(1-t)^{k+1}},$$

for any given $D \ge 1$, provided that C is large enough. In other terms, we need

$$t^N(1-t)^k \le \frac{1}{D}N^{-k}, \text{ for } \frac{1}{2} \le t \le 1-\frac{C}{N},$$

when $C=C(D)$ is large enough and $N \ge N(C) \gg 1$. The left hand side in this inequality is an increasing function of t on the interval $[0, 1/(1+k/N)]$. If $t \le 1-C/N \le 1/(1+k/N)$ (which is fulfilled when $C \ge 2k$ and $N \gg N(C)$) it is

$$\le \left(1-\frac{C}{N}\right)^N \left(\frac{C}{N}\right)^k = \left(1+\mathcal{O}_C\left(\frac{1}{N}\right)\right) e^{-C} C^k N^{-k}.$$

This is $\le N^{-k}/D$ if $C \ge C(D)$, $N \ge N(C)$. □

For simplicity we will restrict the attention to the region

$$|z| \le r_0 - 1/N \le 1-2/N, \tag{4.2.43}$$

where $G \asymp (1-|z|)^{-1}$, $G' \asymp (1-|z|)^{-2}$.

It follows from the calculation (4.3.6) below, that

$$|\partial_z Z|^2 = \left(\frac{2}{t}\left(K(t\partial_t)^2 K + (t\partial_t K)^2\right)\right)_{t=|z|^2}.$$

This is $\asymp 1$ for $|z| \le 1/2$ and for $1/2 \le |z| < 1-1/N$ it is in view of Proposition 4.2.4 and the subsequent observation

$$\asymp M_{N,0}M_{N,2} + M_{N,1}^2 \asymp \frac{1}{(1-t)^4},\ t=|z|^2.$$

In the region (4.2.43) we get:

$$|Z'(z)| \asymp G(|z|)^2. \tag{4.2.44}$$

(4.2.35), (4.2.43), (4.2.44) will be used in (4.2.32), (4.2.33).

Combining the implicit function theorem and Rouché's theorem to (4.2.28),we see that for $|\alpha'| < C_1 N$, $\alpha' = \sum_2^N \alpha_j e_j \in \overline{Z}(z)^\perp$, the equation

$$\widetilde{g}(z, \alpha_1, \alpha') = 0 \tag{4.2.45}$$

has a unique solution

$$\alpha_1 = f(z, \alpha') \in D(0, C_1 N/G(|z|)). \tag{4.2.46}$$

Here, we also use (4.2.20), (4.2.25). Moreover, f satisfies

$$f(z,\alpha') = \frac{z^N}{\delta|Z|^2} + \mathcal{O}(1)\delta N^2 = \mathcal{O}(1)\left(\frac{|z|^N}{\delta G^2} + \delta N^2\right). \tag{4.2.47}$$

Differentiating the equation (4.2.45) (where $\alpha_1 = f$) we get

$$\partial_z \widetilde{g} + \partial_\alpha \widetilde{g}\partial_z f = 0,\ \partial_{\overline{z}}\widetilde{g} + \partial_\alpha \widetilde{g}\partial_{\overline{z}} f = 0.$$

Hence,

$$\begin{cases} \partial_z f = -\left(\partial_{\alpha_1}\widetilde{g}\right)^{-1}\partial_z \widetilde{g}, \\ \partial_{\overline{z}} f = -\left(\partial_{\alpha_1}\widetilde{g}\right)^{-1}\partial_{\overline{z}} \widetilde{g}. \end{cases} \tag{4.2.48}$$

Since $\widetilde{g}$ is holomorphic in α_1, α' and in $\alpha_1, \alpha_2, ..., \alpha_{N^2}$, we see that f is holomorphic in α' and in $\alpha_2, ..., \alpha_{N^2}$ Applying $\partial_{\alpha_2}, ..., \partial_{\alpha_{N^2}}$ to (4.2.45), we get

$$\partial_{\alpha_j} f = -\left(\partial_{\alpha_1}\widetilde{g}\right)^{-1}\partial_{\alpha_j}\widetilde{g},\ 2 \le j \le N^2. \tag{4.2.49}$$

Combining (4.2.29) in the form,

$$\partial_{\alpha_1}\widetilde{g}(z,\alpha) = -(1+\mathcal{O}(G(|z|)\delta N))\delta |Z(z)|^2,$$

(4.2.30), (4.2.32), (4.2.33) with (4.2.48) and (4.2.49), we get

$$\begin{aligned} \partial_z f = &\frac{(1+\mathcal{O}(G\delta N))}{\delta |Z(z)|^2}\times \\ &\left(N z^{N-1} - \delta f \partial_z\left(|Z|^2\right) + \mathcal{O}\left(G^2\delta^2 N\right)\left|\sum_2^{N^2}\alpha_j \partial_z e_j\right| + \mathcal{O}(1)\frac{(G\delta N)^2}{r_0 - |z|}\right). \end{aligned} \tag{4.2.50}$$

$$\begin{aligned} \partial_{\overline{z}} f = &\frac{(1+\mathcal{O}(G\delta N))}{\delta |Z(z)|^2}\times \\ &\left(-\delta f \partial_{\overline{z}}\left(|Z|^2\right) + \mathcal{O}\left(G^2\delta^2 N\right)\left|f\overline{\partial_z Z} + \sum_2^{N^2}\alpha_j \partial_{\overline{z}} e_j\right|\right), \end{aligned} \tag{4.2.51}$$

$$\partial_{\alpha_j} f = \mathcal{O}(1)\frac{G^2\delta^2 N}{\delta G^2} = \mathcal{O}(\delta N),\ 2 \le j \le N^2. \tag{4.2.52}$$

From (4.2.35) and the observation prior to Proposition 4.2.4 we know that

$$\partial_z\left(|Z|^2\right),\ \partial_{\overline{z}}\left(|Z|^2\right) \asymp G(|z|)^3|z|.$$

Recall also that $|Z| \asymp G(|z|)$. Using this in (4.2.50), (4.2.51), we get

$$\begin{aligned} \partial_z f = &\frac{\mathcal{O}(1)}{\delta G^2}\times \\ &\left(N|z|^{N-1} + \delta |f| G^3|z| + \mathcal{O}\left(G^2\delta^2 N\right)\left|\sum_2^{N^2}\alpha_j \partial_z e_j\right| + \mathcal{O}(1)\frac{G^2\delta^2 N^2}{r_0 - |z|})\right). \end{aligned} \tag{4.2.53}$$

4.3 | Choosing appropriate coordinates

The next task will be to choose an orthonormal basis $e_1(z), e_2, ..., e_{N^2}(z)$ in $\mathbb{C}^{N^2}$ with

$$e_1(z) = |Z(z)|^{-1}\overline{Z}(z)$$

such that we get a good control over $\sum_2^{N^2}\alpha_j \partial_z e_j$, $\sum_2^{N^2}\alpha_j \partial_{\overline{z}} e_j$ and such that

$$dQ_1 \wedge ... \wedge dQ_{N^2}|_{\alpha_1 = f(z,\alpha')}$$

can be expressed easily up to small errors. Consider a point $z_0 \in D(0, r_0 - N^{-1})$. We shall see below that the vectors $\overline{Z}(z)$, $\overline{\partial_z Z}(z)$ are linearly independent for every $z \in D(0,1)$

Proposition 4.3.1. *There exists an orthonormal basis $e_1(z), e_2(z), ..., e_{N^2}(z)$ in $\mathbb{C}^{N^2}$, depending smoothly on $z \in$ neigh(z_0) such that*

$$e_1(z) = |Z(z)|^{-1}\overline{Z}(z), \tag{4.3.1}$$

$$\mathbb{C}e_1(z_0) \oplus \mathbb{C}e_2(z_0) = \mathbb{C}\overline{Z}(z_0) \oplus \overline{\partial_z Z}(z_0), \tag{4.3.2}$$

$$e_j(z) - e_j(z_0) = \mathcal{O}((z-z_0)^2),\ \ j \ge 3. \tag{4.3.3}$$

Proof. We choose $e_1(z)$ as in (4.3.1). Let $e_3(z_0),...,e_{N^2}(z_0)$ be an orthonormal basis in

$$\left(\mathbb{C}\overline{Z}(z_0)\oplus\mathbb{C}\overline{\partial_z Z}(z_0)\right)^{\perp}.$$

Then, we get an orthonormal family $e_3(z),...,e_{N^2}(z)$ in $e_1(z)^{\perp}$ in the following way:

Let V_0 be the isometry $\mathbb{C}^{N^2-2}\to\mathbb{C}^{N^2}$, defined by $V_0 v_j^0 = e_j(z_0)$, $j=3,...,N^2$, where $v_3^0,...,v_{N^2}^0$ is the canonical basis in $\mathbb{C}^{N^2-2}$ with a non-canonical labeling. Let $\pi(z)u=(u|e_1(z))e_1(z)$ be the orthogonal projection onto $\mathbb{C}e_1(z)$. For $z\in\mathrm{neigh}(z_0,\mathbb{C})$, let $V(z)=(1-\pi(z))V_0$. Then $f_j(z)=V(z)v_j^0$, $j=3,...,N^2$ form a linearly independent system in $e_1(z)^{\perp}$ and we get an orthonormal system of vectors that span the same hyperplane in $e_1(z)^{\perp}$ by Gram orthonormalization,

$$e_j(z)=V(z)(V^*(z)V(z))^{-\frac{1}{2}}v_j^0,\quad 3\le j\le N^2.$$

We have

$$V(z)v_j^0=(1-\pi(z))e_j(z_0)=e_j(z_0)-(e_j(z_0)|e_1(z))e_1(z),$$

$$(e_j(z_0)|e_1(z))=\frac{(e_j(z_0)|\overline{Z}(z))}{|Z(z)|}=\mathcal{O}((z-z_0)^2),$$

since $(e_j(z_0)|\overline{Z}(z))=e_j(z_0)\cdot Z(z)=:k(z)$ is a holomorphic function of z with

$$k(z_0)=(e_j(z_0)|\overline{Z}(z_0))=0,\quad k'(z_0)=(e_j(z_0)|\overline{\partial_z Z}(z_0))=0.$$

Thus, $V(z)=V(z_0)+\mathcal{O}(z-z_0)^2)$ and we conclude that (4.3.3) holds. Let $e_2(z)$ be a normalized vector in $(e_1(z),e_3(z),e_4(z),...,e_{N^2}(z))^{\perp}$ depending smoothly on z. Then $e_1(z),e_2(z),...,e_{N^2}(z)$ is an orthonormal basis and since $e_3(z_0),...,e_{N^2}(z_0)$ are orthogonal to $\overline{Z}(z_0),\overline{\partial Z}(z_0)$ by construction, we get (4.3.2). □

We can make the following explicit choice:

$$e_2(z)=|f_2|^{-1}f_2,\ f_2=\overline{\partial_z Z}(z)-\sum_{j\neq 2}(\overline{\partial_z Z}(z)|e_j(z))e_j(z),\tag{4.3.4}$$

so that for $z=z_0$,

$$e_2(z_0)=|f_2(z_0)|^{-1}f_2(z_0),\ f_2(z_0)=\overline{\partial_z Z}(z_0)-(\overline{\partial_z Z}(z_0)|e_1(z_0))e_1(z_0).\tag{4.3.5}$$

We next compute some scalar products and norms with Z and $\partial_z Z$. Recall that

$$Z(z)=\left(z^{N-j+k-1}\right)_{j,k=1}^{N}$$

and that $|Z(z)|=K(|z|^2)$, $K(t)=\sum_0^{N-1}t^{\nu}$. Repeating basically the same computation, we get

$$z\partial_z Z=\left((N-j+k-1)z^{N-j+k-1}\right)_{j,k=1}^{N},$$

and

$$\begin{aligned}|z\partial_z Z|^2&=\sum_{j,k=1}^{N}(N-j+k-1)^2|z|^{2(N-j+k-1)}=\sum_{\nu,\mu=0}^{N-1}(\nu+\mu)^2|z|^{2(\nu+\mu)}\\&=\sum_0^{N-1}\nu^2|z|^{2\nu}\sum_0^{N-1}|z|^{2\mu}+2\sum_0^{N-1}\nu|z|^{2\nu}\sum_0^{N-1}\mu|z|^{2\mu}+\sum_0^{N-1}|z|^{2\nu}\sum_0^{N-1}\mu^2|z|^{2\mu}\\&=2\left(K(t\partial_t)^2K+(t\partial_t K)^2\right)_{t=|z|^2}.\end{aligned}\tag{4.3.6}$$

Similarly,

$$\begin{aligned}(z\partial_z Z|Z) &= \sum_{j,k=1}^{N}(N-j+k-1)|z|^{2(N-j+k-1)}\\ &= \sum_{\nu=0}^{N-1}\sum_{\mu=0}^{N-1}(\nu+\mu)|z|^{2(\nu+\mu)}\\ &=2(Kt\partial_t K)_{t=|z|^2}.\end{aligned}$$

Then, by a straight forward calculation,

$$|\partial_z Z|^2-\frac{|(\partial_z Z|Z)|^2}{|Z|^2}=\left(\frac{2}{t}\left(K(t\partial_t)^2K-(t\partial_t K)^2\right)\right)_{t=|z|^2} \tag{4.3.7}$$

Here,

$$\begin{aligned}\frac{2}{t}\left(K(t\partial_t)^2K-(t\partial_t K)^2\right)&=\frac{2}{t}\sum_0^{N-1}t^\nu\sum_0^{N-1}\nu^2t^\nu-\frac{2}{t}\left(\sum_0^{N-1}\nu t^\nu\right)^2\\ =\sum_{\nu,\mu=0}^{N-1}\left(\nu^2+\mu^2-2\nu\mu\right)t^{\nu+\mu-1}&=\sum_{\nu,\mu=0}^{N-1}(\nu-\mu)^2t^{\nu+\mu-1}\\ &=\sum_{k=0}^{2N-3}a_{k,N}t^k,\end{aligned}$$

where

$$a_{k,N}=\sum_{\substack{\nu+\mu-1=k\\ 0\le\nu,\mu\le N-1}}(\nu-\mu)^2.$$

We observe that

$$a_{k,N}\le\mathcal{O}(1)(1+k)^3 \text{ uniformly with respect to } N,$$
$$a_{k,N}=a_{k,\infty} \text{ is independent of } N \text{ for } k\le N-2,$$
$$a_{k,\infty}\ge(1+k)^3/\mathcal{O}(1).$$

We conclude that

$$\frac{1}{C}\left(1+M_{N-1,3}\right)\le\frac{2}{t}\left(K(t\partial_t)^2K-(t\partial_t K)^2\right)\le C\left(1+M_{2N-2,3}\right)$$

and (4.2.40) shows that the first and third members are of the same order of magnitude,

$$\asymp 1+M_{N,3}(t)\asymp\min\left(\frac{1}{1-t},N\right)^4$$

which is $\asymp 1+M_{\infty,3}(t)$, for $0\le t\le 1-1/N$. From this and Proposition 4.2.4 we get:

Proposition 4.3.2. *We have*

$$\frac{2}{t}(K(t\partial_t)^2K-(t\partial_t K)^2)\asymp K^4,\ 0<t\le 1-1/N, \tag{4.3.8}$$

where we recall that $K=K_N$ depends on N (cf. (4.2.34)) and that

$$K_N=K_\infty-\frac{t^N}{1-t}.$$

We have

$$\begin{cases} t\partial_t K_N=t\partial_t K_\infty+\mathcal{O}\left(\frac{Nt^N}{1-t}\right), & t\le 1-\frac{1}{N},\\ (t\partial_t)^2K_N=(t\partial_t)^2K_\infty+\mathcal{O}\left(\frac{N^2t^N}{1-t}\right), & t\le 1-\frac{1}{N},\end{cases} \tag{4.3.9}$$

and it follows that

$$\begin{aligned}&\frac{2}{t}\left(K_N(t\partial_t)^2K_N-(t\partial_tK_N)^2\right)-\frac{2}{t}\left(K_\infty(t\partial_t)^2K_\infty-(t\partial_tK_\infty)^2\right)\\&=\mathcal{O}\left(\frac{N^2t^N}{(1-t)^2}\right),\end{aligned}\tag{4.3.10}$$

for $t\le 1-1/N$.

Proposition 4.3.2 and (4.3.7) give

$$|\partial_zZ|^2-\frac{|(\partial_zZ|Z)|^2}{|Z|^2}\asymp K(|z|^2)^4.\tag{4.3.11}$$

This implies that ∂_zZ, Z are linearly independent.

Assume that

$$|\nabla_ze_1(z)|=\mathcal{O}(m)$$

for some weight $m\ge 1$. We shall see below that this holds when $m=K(|z|^2)$. Then $\|\nabla_z\Pi\|=\mathcal{O}(m)$ and hence $\|\nabla_zV\|=\mathcal{O}(m)$. It follows that $\|\nabla_z(V^*(z)V(z))\|=\mathcal{O}(m)$. By standard (Cauchy-Riesz) functional calculus, using also that $\|V(z)^{-1}\|=\mathcal{O}(1)$, we get $\|\nabla_z(V^*(z)V(z))^{-\frac{1}{2}}\|=\mathcal{O}(m)$. Hence $\|\nabla_zU(z)\|=\mathcal{O}(m)$, where

$$U(z)=V(z)(V^*(z)V(z))^{-1/2}$$

is the isometry appearing in the proof of Proposition 4.3.1. Since $\nabla_ze_j=(\nabla_zU(z))\nu_j^0$, we conclude that $\|\nabla_zU(z)\|=\mathcal{O}(m)$, so

$$\left|\sum_3^{N^2}\alpha_j\nabla_ze_j\right|\le\mathcal{O}(m)\|\alpha\|_{\mathbb{C}^{N^2-2}}.\tag{4.3.12}$$

We next show that we can take $m=K(|z|^2)$. We have

$$\nabla_ze_1=\frac{\nabla_z\overline{Z}}{|Z|}-\frac{\nabla_z|Z|}{|Z|^2}\overline{Z}=\frac{\nabla_z\overline{Z}}{K}-\frac{K'\nabla_z(z\overline{z})}{K^2}\overline{Z}.\tag{4.3.13}$$

By (4.3.6),

$$|\partial_zZ|=\left(\frac{2}{t}\left(K(t\partial_t)^2K+(t\partial_tK)^2\right)\right)^{\frac{1}{2}}_{t=|z|^2}=\mathcal{O}(K^2).$$

Since Z is holomorphic, this leads to the same estimates for $|\nabla_zZ|$ and $|\nabla_z\overline{Z}|$, and $|\partial_z^2Z|=\mathcal{O}(K^3)$, for $|z|<1-N^{-1}$, by the Cauchy inequalities. Using this in (4.3.13), we get

$$|\nabla_ze_1|=\mathcal{O}(K).\tag{4.3.14}$$

Thus we can take $m=K(|z|^2)$ in (4.3.12). Let f_2 be the vector in (4.3.4) so that $e_2(z)=|f_2|^{-1}f_2$. Recall that $e_j=U(z)\nu_j^0$, where we now know that $\|\nabla_zU(z)\|=\mathcal{O}(K)$. Write,

$$\nabla_zf_2=\nabla_z\overline{\partial_zZ}-\sum_{j\ne 2}\left((\nabla_z\overline{\partial_zZ}|e_j)e_j+(\overline{\partial_zZ}|\nabla_ze_j)e_j+(\overline{\partial_zZ}|e_j)\nabla_ze_j\right).$$

Here, $|\nabla_z\overline{\partial_zZ}|=\mathcal{O}(K^3)$, as we have just seen. It is also clear that the term for $j=1$ in the sum above is $\mathcal{O}(K^3)$. It remains to study $|\mathrm{I}+\mathrm{II}+\mathrm{III}|\le|\mathrm{I}|+|\mathrm{II}|+|\mathrm{III}|$, where

$$\begin{aligned}\mathrm{I}&=\sum_3^{N^2}(\nabla_z\overline{\partial_zZ}|e_j)e_j,\\\mathrm{II}&=\sum_3^{N^2}(\overline{\partial_zZ}|\nabla_ze_j)e_j,\\\mathrm{III}&=\sum_3^{N^2}(\overline{\partial_zZ}|e_j)\nabla_ze_j.\end{aligned}$$

Here, $|\mathrm{I}| \le |\nabla_z \overline{\partial_z Z}| = \mathcal{O}(K^3)$ and by (4.3.12) we have $|\mathrm{III}| \le \mathcal{O}(K)|\overline{\partial_z Z}| = \mathcal{O}(K^3)$. Further,

$$\begin{aligned}\mathrm{II} &= \sum_3^{N^2} (\overline{\partial_z Z}|(\nabla_z U(z))\nu_j^0)e_j \\ &= \sum_3^{N^2} ((\nabla_z U(z))^* \overline{\partial_z Z}|\nu_j^0)e_j,\end{aligned}$$

so

$$|\mathrm{II}| = |(\nabla_z U(z))^* \overline{\partial_z Z}| = \mathcal{O}(K)K^2 = \mathcal{O}(K^3).$$

Thus,

$$|\nabla_z f_2| = \mathcal{O}(K^3). \tag{4.3.15}$$

Recall from (4.3.5) that for $z = z_0$,

$$\begin{aligned} f_2 &= \overline{\partial_z Z} - (\overline{\partial_z Z}|e_1)e_1, \\ |f_2|^2 &= |\partial_z Z|^2 - \frac{|(\partial_z Z|Z)|^2}{|Z|},\end{aligned}$$

so by (4.3.11),

$$|f_2(z_0)| \asymp K(|z_0|^2)^2,$$

Hence,

$$|f_2(z)| \asymp K^2,\ z \in \mathrm{neigh}\,(z_0).$$

From this, (4.3.4) and (4.3.11), we conclude first that $\nabla_z |f_2| = \mathcal{O}(K^3)$ and then that

$$|\nabla_z e_2| = \mathcal{O}(K). \tag{4.3.16}$$

This completes the proof of the fact that we can take $m = K$ above. In particular (4.3.12) holds with $m = K(|z|^2) \asymp G(|z|)$, so

$$\left| \sum_2^{N^2} \alpha_j \partial_z e_j \right| \le \mathcal{O}(1) G|\alpha| \le \mathcal{O}(1) GN, \tag{4.3.17}$$

where we used the assumption that $|Q| \le C_1 N$ in the last step.

Combining this with (4.2.53), (4.2.52), (4.2.47), (4.2.35) and the observation prior to Proposition 4.2.4, we get

$$\begin{aligned}\partial_z f &= \frac{\mathcal{O}(1)}{\delta G^2}\left(N|z|^{N-1} + \delta\left(\frac{|z|^N}{\delta G^2} + \delta N^2\right)G^3 + G^2\delta^2 NGN + \frac{G^2\delta^2 N^2}{r_0 - |z|}\right) \\ &= \mathcal{O}(1)\left(\frac{N|z|^{N-1}}{\delta G^2} + \frac{|z|^N}{\delta G} + G\delta N^2 + \frac{\delta N^2}{r_0 - |z|}\right).\end{aligned}$$

In the last parenthesis the second term is dominated by the first one and the third term is dominated by the fourth. If we recall that $r_0 - |z| \ge 1/N$, we get

$$\partial_z f = \mathcal{O}(1)\left(\frac{N|z|^{N-1}}{\delta G^2} + \delta N^3\right). \tag{4.3.18}$$

Similarly, from (4.2.51), (4.2.44) we get

$$\begin{aligned}\partial_{\overline{z}} f &= \frac{\mathcal{O}(1)}{\delta G^2}\left(\delta\left(\frac{|z|^N}{\delta G^2} + \delta N^2\right)G^3 + G^2\delta^2 N\left(\left(\frac{|z|^N}{\delta G^2} + \delta N^2\right)G^2 + GN\right)\right) \\ &= \mathcal{O}(1)\left(\frac{|z|^N}{\delta G} + \delta N^2 G + N|z|^N + G^2\delta^2 N^3 + G\delta N^2\right).\end{aligned}$$

Using (4.2.20), we get

$$\partial_{\overline{z}} f = \mathcal{O}(1)\left(\frac{|z|^N}{\delta G} + \delta N^2 G\right), \tag{4.3.19}$$

see (4.2.47). This will be used together with the estimates $\partial_{\alpha_j} f = \mathcal{O}(\delta N)$ in (4.2.52).

Proposition 4.3.3. *We express Q in the canonical basis in $\mathbb{C}^{N^2}$ or in any other fixed orthonormal basis. Let $e_1(z),\dots,e_{N^2}(z)$ be an orthonormal basis in $\mathbb{C}^{N^2}$ depending smoothly on z and with $e_1(z) = |Z(z)|^{-1}\overline{Z}(z)$, $\mathbb{C}e_1(z)\oplus\mathbb{C}e_2(z) = \mathbb{C}\overline{Z}(z)\oplus\overline{\partial_z Z}(z)$. Write $Q = \alpha_1\overline{Z}(z)+\sum_2^{N^2}\alpha_j e_j(z)$, and recall that the hypersurface*

$$\{(z,Q)\in D(0,r_0-1/N)\times B(0,C_1N);\, E^{\delta}_{-+}(z,Q)=0\}$$

is given by (4.2.46) with f as in (4.2.47). Then the restriction of $dQ\wedge d\overline{Q}$ to this hypersurface, is given by

$$\begin{aligned}
&dQ\wedge d\overline{Q} = J(f)\,dz\wedge d\overline{z}\wedge d\alpha'\wedge d\overline{\alpha}',\\
&J(f) = -\frac{|\alpha_2|^2}{|Z|^2}\left|\left(e_2|\overline{\partial_z Z}\right)\right|^2+\mathcal{O}(1)\left(\frac{N|z|^{N-1}}{\delta G}+G\delta N^3+|\alpha_2|\delta NG^2\right)^2\\
&\qquad +\mathcal{O}(1)|\alpha_2|G\left(\frac{N|z|^{N-1}}{\delta G}+G\delta N^3+|\alpha_2|G^2\delta N\right).
\end{aligned}\tag{4.3.20}$$

Here $\alpha' = (\alpha_2,\dots,\alpha_{N^2})$, $d\alpha'\wedge d\overline{\alpha}' = d\alpha_2\wedge d\overline{\alpha}_2\wedge\dots\wedge d\alpha_{N^2}\wedge d\overline{\alpha}_{N^2}$.

Proof. The differential form $dQ_1\wedge dQ_2\wedge\dots\wedge dQ_{N^2}$ will change only by a factor of modulus one if we express Q in another fixed orthonormal basis and we will choose for that the basis $e_1(z_0),\dots,e_{N^2}(z_0)$:

$$Q=\sum_1^{N^2}Q_k e_k(z_0),\quad Q_k=(Q|e_k(z_0)).$$

Write

$$Q=\alpha_1\underbrace{\overline{Z}(z)}_{|Z(z)|e_1(z)}+\sum_2^{N^2}\alpha_k e_k(z)$$

and restrict to $\alpha_1 = f(z,\alpha_2,\dots,\alpha_{N^2})$, where we sometimes identify $\alpha'\in\overline{Z}(z)^{\perp}$ with $(\alpha_2,\dots,\alpha_{N^2})$:

$$Q_{|\alpha_1=f(z,\alpha')} = f(z,\alpha')\overline{Z}(z)+\sum_2^{N^2}\alpha_k e_k(z).$$

Then,

$$\begin{aligned}
Q_j &= f(\overline{Z}(z)|e_j(z_0))+\sum_{k=2}^{N^2}\alpha_k(e_k(z)|e_j(z_0)),\\
dQ_j &=(d_zf+d_{\alpha'}f)(\overline{Z}(z)|e_j(z_0))+f(d_z\overline{Z}(z)|e_j(z_0))\\
&\quad+\sum_{k=2}^{N^2}\alpha_k(d_ze_k(z)|e_j(z_0))+\sum_{k=2}^{N^2}d\alpha_k(e_k(z)|e_j(z_0)).
\end{aligned}$$

Taking $z=z_0$ until further notice, we get with $\alpha'=(\alpha_2,\dots,\alpha_{N^2})$:

$$dQ_j = (d_zf+d_{\alpha'}f)(\overline{Z}|e_j)+f(\overline{\partial_z Z}|e_j)d\overline{z}+\alpha_2(d_ze_2|e_j)+\begin{cases}d\alpha_j,\ j\geq 2,\\ 0,\ j=1\end{cases}.$$

Here, we used (4.3.3). The first term to the right is equal to $(d_zf+d_{\alpha'}f)|\overline{Z}|$ when $j=1$ and it vanishes when $j\geq 2$. The second term vanishes for $j\geq 3$, by (4.3.2). The third term is equal to $-\alpha_2(e_2|d_ze_j)$ (by differentiation of the identity $(e_2|e_j)=\delta_{2,j}$) and it vanishes for $j\geq 3$ (remember that we take $z=z_0$). Thus, for $z=z_0$:

$$\begin{aligned}
dQ_1 &= |\overline{Z}|(d_zf+d_{\alpha'}f)+f(\overline{\partial_z Z}|e_1)d\overline{z}-\alpha_2(e_2|d_ze_1),\\
dQ_2 &= d\alpha_2+f(\overline{\partial_z Z}|e_2)d\overline{z}-\alpha_2(e_2|d_ze_2),\\
dQ_j &= d\alpha_j,\ j\geq 3.
\end{aligned}$$

When forming $dQ_1\wedge d\overline{Q}_1\wedge\dots\wedge dQ_{N^2}\wedge d\overline{Q}_{N^2}$ we see that the terms in $d\alpha_j$ for $j\geq 3$ in the expression for dQ_1 will not contribute, so in that expression we can replace $d_{\alpha'}f$ by $\partial_{\alpha_2}f\,d\alpha_2$. Using (4.3.18), (4.3.19), (4.2.52), (4.2.47), (4.2.44) this gives, where "≡" means equivalence up to terms that do not influence the $2N^2$ form above:

$$\begin{aligned} dQ_1 \equiv &-\alpha_2(e_2|d_z e_1) + \mathscr{O}(1)\left(\frac{N|z|^{N-1}}{\delta G} + G\delta N^3\right)dz \\ &+ \mathscr{O}(1)\left(\frac{|z|^N}{\delta} + G^2\delta N^2\right)d\overline{z} + \mathscr{O}(\delta N G)d\alpha_2. \end{aligned}$$

Similarly, using also (4.3.16),

$$dQ_2 = d\alpha_2 + \mathscr{O}\left(\frac{|z|^N}{\delta} + \delta N^2 G^2 + |\alpha_2|G\right)d\overline{z} + \mathscr{O}\left(|\alpha_2|G\right)dz.$$

When computing $dQ_1 \wedge dQ_2$ we notice that the terms in $dz \wedge d\overline{z}$ will not contribute to the $2N^2$-form $dQ_1 \wedge d\overline{Q}_1 \wedge ... \wedge dQ_{N^2} \wedge d\overline{Q}_{N^2}$. We get

$$\begin{aligned} dQ_1 \wedge dQ_2 \equiv &-\alpha_2(e_2|d_z e_1) \wedge d\alpha_2 \\ &+ \mathscr{O}(1)\left(\frac{N|z|^{N-1}}{\delta G} + G\delta N^3 + |\alpha_2|\delta N G^2\right)dz \wedge d\alpha_2 \\ &+ \mathscr{O}(1)\left(\frac{|z|^N}{\delta} + G^2\delta N^2 + |\alpha_2|\delta N G^2\right)d\overline{z} \wedge d\alpha_2. \end{aligned} \tag{4.3.21}$$

Here,

$$\begin{aligned} (e_2|d_z e_1) &= \left(e_2|d_z\left(|Z|^{-1}\right)\overline{Z}\right) = \left(e_2||Z|^{-1}d_z\overline{Z}\right) + \left(e_2|d_z\left(|Z|^{-1}\right)\overline{Z}\right) \\ &= |Z|^{-1}\left(e_2|\overline{\partial_z Z d\overline{z}}\right) + 0 = |Z|^{-1}(e_2|\overline{\partial_z Z})dz, \end{aligned}$$

so the first term in (4.3.21) is equal to

$$-\frac{\alpha_2}{|Z|}(e_2|\overline{\partial_z Z})dz \wedge d\alpha_2 = \mathscr{O}(1)\alpha_2 G dz \wedge d\alpha_2.$$

Notice that $dQ_1 \wedge d\overline{Q}_1 \wedge dQ_2 \wedge d\overline{Q}_2 = -dQ_1 \wedge dQ_2 \wedge d\overline{Q}_1 \wedge d\overline{Q}_2$. From (4.3.21) and its complex conjugate we get

$$\begin{aligned} &dQ_1 \wedge d\overline{Q}_1 \wedge dQ_2 \wedge d\overline{Q}_2 \\ &\equiv \left(-\frac{|\alpha_2|^2}{|Z|^2}\left|\left(e_2|\overline{\partial_z Z}\right)\right|^2 + \mathscr{O}(1)\left(\frac{N|z|^{N-1}}{\delta G} + G\delta N^3 + |\alpha_2|\delta N G^2\right)^2\right. \\ &\left.+\mathscr{O}(1)|\alpha_2|G\left(\frac{N|z|^{N-1}}{\delta G} + G\delta N^3 + |\alpha_2|G^2\delta N\right)\right)dz \wedge d\overline{z} \wedge d\alpha_2 \wedge d\overline{\alpha}_2. \end{aligned}$$

□

4.4 | Proof of Theorem 1.4.3

Let $Q \in \mathbb{C}^{N^2}$ be an $N \times N$ matrix whose entries are independent random variables $\sim \mathscr{N}_{\mathbb{C}}(0,1)$, so that the corresponding probability measure is

$$\pi^{-N^2}e^{-|Q|^2}(2i)^{-N^2}d\overline{Q}_1 \wedge dQ_1 \wedge ... \wedge d\overline{Q}_{N^2} \wedge dQ_{N^2} = \frac{1}{(2\pi i)^{N^2}}e^{-|Q|^2}d\overline{Q} \wedge dQ.$$

We are interested in

$$K_\phi = \mathbb{E}\left(1_{B_{\mathbb{C}^{N^2}}}(0,1)\sum_{\lambda \in \sigma(A_0+\delta Q)}\phi(\lambda)\right), \quad \phi \in C_0(D(0, r_0 - 1/N)), \tag{4.4.1}$$

which is of the form (4.1.3) with

$$m(Q) = 1_{B_{\mathbb{C}^{N^2}}}(Q)\pi^{-N^2}e^{-|Q|^2}, \tag{4.4.2}$$

so we have (4.1.8), (4.1.9) with $J(f)$ as in (4.3.20) and f as in (4.2.46). More explicitly,

$$\widetilde{\widetilde{\Xi}}(z) = \int_{|f|^2|Z(z)|^2+|\alpha'|^2\le C_1^2N^2} \pi^{-N^2}e^{-|f(z,\alpha')|^2|Z(z)|^2-|\alpha'|^2}J(f)(z,\alpha')L(d\alpha').$$

By (4.2.47), (4.2.20), (4.2.25) :

$$|f| \le \mathcal{O}(1)\frac{N}{G}\left(\frac{|z|^N}{\delta NG}+\delta NG\right) \ll \frac{N}{G}.$$

We now strengthen (4.2.20), (4.2.25) to the assumption

$$\frac{|z|^N}{\delta NG}+\delta NG \ll \frac{1}{N}, \text{ for all } z\in D(0,r_0), \tag{4.4.3}$$

implying that $|f|G \ll 1$, for all $z \in D(0,r_0)$. Equivalently, by the same reasoning as after (4.2.26), r_0 should satisfy

$$\frac{r_0^N}{\delta NG(r_0)}+\delta NG(r_0) \ll \frac{1}{N}. \tag{4.4.4}$$

Then

$$e^{-|f(z,\alpha')|^2|Z(z)|^2} = 1+\mathcal{O}(1)N^2\left(\frac{|z|^N}{\delta NG}+\delta NG\right)^2,$$

and using (4.3.20), we get

$$\begin{aligned}\widetilde{\widetilde{\Xi}}(z) = &\left(1+\mathcal{O}(1)N^2\left(\frac{|z|^N}{\delta NG}+\delta NG\right)^2\right)\times \\ &\frac{|(e_2|\overline{\partial_z Z})|^2}{|Z|^2}\int_{|(f|Z|,\alpha')|\le C_1N}|\alpha_2|^2e^{-|\alpha'|^2}\pi^{1-N^2}L(d\alpha') \\ &+\mathcal{O}(1)\int e^{-|\alpha'|^2}\left(\frac{N|z|^{N-1}}{\delta G}+G\delta N^3+|\alpha_2|\delta NG^2\right)^2\pi^{1-N^2}L(d\alpha') \\ &+\mathcal{O}(1)\int e^{-|\alpha'|^2}|\alpha_2|G\left(\frac{N|z|^{N-1}}{\delta G}+G\delta N^3+|\alpha_2|\delta NG^2\right)\pi^{1-N^2}L(d\alpha').\end{aligned}$$

Since $|f||Z| \ll N$, the first integral is equal to

$$\int_{\mathbb{C}}\frac{1}{\pi}|w|^2e^{-|w|^2}L(dw)+\mathcal{O}\left(e^{-N^2/\mathcal{O}(1)}\right) = 1+\mathcal{O}\left(e^{-N^2/\mathcal{O}(1)}\right).$$

The sum of the other two integrals is equal to

$$\begin{aligned}&\mathcal{O}(1)\left(\left(\frac{N|z|^{N-1}}{\delta G}+G\delta N^3+\delta NG^2\right)^2+G\left(\frac{N|z|^{N-1}}{\delta G}+G\delta N^3+\delta NG^2\right)\right) \\ &=\mathcal{O}(1)\left(\left(\frac{N|z|^{N-1}}{\delta G}+G\delta N^3\right)^2+G\left(\frac{N|z|^{N-1}}{\delta G}+G\delta N^3\right)\right).\end{aligned}$$

Noticing that

$$\frac{|(e_2|\overline{\partial_z Z})|^2}{|Z|^2} = \mathcal{O}(G^2),$$

we deduce that

$$\begin{aligned}\widetilde{\widetilde{\Xi}}(z) = &\frac{|(e_2|\overline{\partial_z Z})|^2}{|Z|^2} \\ &+\mathcal{O}(1)\left(G^2N^2\left(\frac{|z|^{N-1}}{\delta G^2}+\delta N^2\right)^2+G^2N\left(\frac{|z|^{N-1}}{\delta G^2}+\delta N^2\right)\right).\end{aligned} \tag{4.4.5}$$

We next study the leading term in (4.4.5), given by

$$\frac{|(\overline{\partial_z Z}|e_2)|^2}{\pi|Z|^2}. \tag{4.4.6}$$

Since $\overline{\partial_z Z}$ belongs to the span of $e_1 = \overline{\partial_z Z}/|Z|$ and e_2, we have

$$|(\overline{\partial_z Z}|e_2)|^2 = |\overline{\partial_z Z}|^2 - |(\overline{\partial_z Z}|e_1)|^2,$$

so the leading term (4.4.6) is

$$\frac{1}{\pi|Z|^2}\left(|\overline{\partial_z Z}|^2 - \frac{|(\overline{\partial_z Z}|\overline{Z})|^2}{|Z|^2}\right),$$

which by (4.3.7) is equal to

$$\frac{2}{\pi t}\left(\frac{(t\partial_t)^2 K}{K} - \frac{(t\partial_t K)^2}{K^2}\right)_{t=|z|^2}. \tag{4.4.7}$$

Here, $K = K_N(t) = \sum_0^{N-1} t^\nu$ is the function appearing in Proposition 4.3.2. Let us first compute the limiting quantity obtained by replacing $K = K_N$ in (4.4.7) by $K_\infty = 1/(1-t)$. Since $\partial_t K_\infty = K_\infty^2$, we get

$$t\partial_t K_\infty = tK_\infty^2, \ \ (t\partial_t)^2 K_\infty = tK_\infty^2 + 2t^2K_\infty^3,$$

and

$$\frac{2}{\pi t}\left(\frac{(t\partial_t)^2 K_\infty}{K_\infty} - \frac{(t\partial_t K_\infty)^2}{K_\infty^2}\right) = \frac{2}{\pi}K_\infty^2 = \frac{2}{\pi}\frac{1}{(1-t)^2}. \tag{4.4.8}$$

We next approximate the expression (4.4.7) with (4.4.8), using (4.3.10) and the fact that $K = (1+\mathcal{O}(t^N))K_\infty$ (uniformly with respect to N). The expression (4.4.7) is equal to

$$\begin{aligned}
&\frac{2}{\pi t K^2}(K(t\partial_t)^2K - (t\partial_t K)^2) \\
&= \frac{2(1+\mathcal{O}(t^N))}{\pi t K_\infty^2}\left(K_\infty (t\partial_t)^2 K_\infty - (t\partial_t K_\infty)^2 + \mathcal{O}(N^2 t^N K_\infty^2)\right).
\end{aligned}$$

Here,

$$(t\partial_t K_\infty)^2 = \mathcal{O}(t^2K_\infty^4), \ \ K_\infty(t\partial_t)^2K_\infty = \mathcal{O}(tK_\infty^3 + t^2K_\infty^4),$$

so the last expression becomes,

$$\frac{2}{\pi t}\left(\frac{(t\partial_t)^2 K_\infty}{K_\infty} - \frac{(t\partial_t K_\infty)^2}{K_\infty^2}\right) + \mathcal{O}(t^N K_\infty + t^{N+1}K_\infty^2 + t^{N-1}N^2),$$

where the first two terms in the remainder are dominated by the last one. We conclude that the difference between the expressions (4.4.7) and (4.4.8) is $\mathcal{O}(t^{N-1}N^2)$, and using also (4.4.5), we get,

$$\begin{aligned}
\tilde{\Xi}(z) = &\frac{2}{\pi(1-|z|^2)^2} + \mathcal{O}(|z|^{2(N-1)}N^2) \\
&+ \mathcal{O}(1)\left(G^2N^2\left(\frac{|z|^{N-1}}{\delta G^2} + \delta N^2\right)^2 + G^2N\left(\frac{|z|^{N-1}}{\delta G^2} + \delta N^2\right)\right).
\end{aligned} \tag{4.4.9}$$

The remainder term can be written

$$\mathcal{O}(G^2)\left(\frac{|z|^{2(N-1)}N^2}{G^2} + \frac{|z|^{2(N-1)}N^2}{\delta^2G^4} + \delta^2N^6 + \frac{|z|^{N-1}N}{\delta G^2} + \delta N^3\right).$$

By (4.4.3), $\frac{1}{\delta G} \gg N^2$, so the second term is

$$\gg \frac{|z|^{2(N-1)}N^2}{G^2}N^4,$$

which is much larger than the first term. We now strengthen (4.4.3) to

$$\frac{|z|^{N-1}}{\delta G^2} + \delta N^2 \ll \frac{1}{N},$$

or equivalently to

$$\frac{|z|^{N-1} N}{\delta G^2} + \delta N^3 \ll 1. \tag{4.4.10}$$

Then remainder in (4.4.9) becomes

$$\mathscr{O}(G^2)\left(\frac{|z|^{N-1} N}{\delta G^2} + \delta N^3\right),$$

and (4.4.9) becomes

$$\widetilde{\Xi}(z) = \frac{2}{\pi(1-|z|^2)^2}\left(1 + \mathscr{O}\left(\frac{|z|^{N-1} N}{\delta G^2} + \delta N^3\right)\right). \tag{4.4.11}$$

Setting $\widetilde{\Xi} = \frac{1}{2\pi}\Xi$ concludes the proof of Theorem 1.4.3.

APPENDIX A

CODE FOR NUMERICAL SIMULATIONS

A.1 | Simulations for Hager's model

A.1.1 – Plotting eigenvalues

To produce Figure 1.1, the left hand side of Figure 1.6 (resp. 1.2) and the left hand side of Figure 1.8 (resp. 1.3) we used the following MATLAB code with the values of N, h and δ specified in the caption of the Figures.

```
%Discretization of hD+exp(-ix) on the Fourier side
clear global;
clear;

%Initialize Variables
N=0; % number of non zero eigenvalues of D
h=0; % Semiclassical parameter
delta=0; % Perturbation coupling constant
choice=0;
D=0; %Matrix dimension

%Input Parameters
N=input('value of N? ')
h=input('value of h? ')

%Choice in terms of Delta
choice = input('Do you want to set a value for delta?\nIf yes, insert the
     value for delta, if you want delta=exp(-1/h) insert 0')

%Set delta according to choice
if (choice == 0 )
    delta=exp(-1/h); %h not too small, e.g. h=0.05
elseif (choice ~= 0)
    delta = choice;
else
    'error'
    break
end

%Calculate Dimension
D=2*N+1;

%Define Matrices
```

```
HD=sparse(1:(D),1:(D),(1:(D))-N-1);
HC=sparse(1:2*N,2:D,1,D,D);
H=h*HD+HC;

%Random Gaussian matrix, complex Gaussian, Expectation 0, Variance 1
H=H+delta*(1/sqrt(2))*(randn(D)+1i*randn(D));

%Calculate eigenvalues
E=eigs(H);

%Plot
fig=figure;
plot(E,'.')
filename='EigSim';
axis equal;
print(fig,'-dpdf',filename);
```

A.1.2 – Numerical density

Figure 1.7 and the right hand side of Figure 1.2 give a comparison between the theoretical (obtained in Theorem 1.2.12) and experimental average density and average integrated density of eigenvalues of a random perturbation of the discretization of $hD + e^{-ix}$ with the coupling constant δ being polynomially small in h. These figures were obtained using the following MATLAB code:

```
%Discretization of hD+exp(-ix) on the Fourier side
%This program will plot the experimental average density and the
%average integrated density
clear global;
clear;

tic %Time
% Calculate Dimension
N=1999;
D=2*N+1;

h=2*10^(-3); % Semiclassical parameter
delta=2*10^(-12);    % Perturbation coupling constant
nsamp=400;  % Number of Runs
k=1;        % counter

E=[];
E1=[];
TempEigVals=[];
boxlength = h/3;
BoxVec = [0:boxlength:1];
nBox = length(BoxVec);
LengthVec =[0];
NumRealEigVals = [0];
IntEigVals = [0];
L=[0];
HR=[]; %Pertrubed Matrix

%Define unperturbed semiclassical Matrix
H=h*sparse(1:(D),1:(D),(1:(D))-N-1) + sparse(1:2*N,2:D,1,D,D);

% Experiments
for k = 1:(nsamp)
```

```
    HR=H+delta*(1/sqrt(2))*(randn(D)+1i*randn(D)); %Random Gaussian
   matrix, Exp 0, Var 1
    E=eig(HR); %find eigenvalues
    HR=[];
    E1 = [real(E) imag(E)]; % Store temporary

    %Restriction of real part and imaginary part
 for j = 1:length(E1)
         if (abs(E1(j,1))<= 2)
             if (E1(j,2)>= 0)
             TempEigVals=[TempEigVals;[E1(j,1),E1(j,2)]];
             end
         end
 end% END LOOP

E1 = TempEigVals;
TempEigVals =[];

%Sorting
[B,I] = sort(E1(:,2));
%R=real(EigVals);
%EigVals = [R(I) B];

E1=[];

% HistogramData
m=1;
for l = 1:nBox
    if ( m < length(B) )
      while ( B(m)<=BoxVec(l) )
            if (m < length(B) )
            m=m+1;
            else
                break
            end
      end
     LengthVec(l) = m;
    else
     LengthVec(l) = m;
    end
end

for j = 1:(nBox-1)
    if (LengthVec(j+1) ~= 0)
        L(j) = (4*boxlength)^(-1)*(LengthVec(j+1)-LengthVec(j));
    else
        L(j) = 0;
    end
end

%Add number of Eigvals to old, normalized by total Mass in Strip
IntEigVals = IntEigVals + (1/length(B))*LengthVec;
NumRealEigVals = NumRealEigVals + (1/length(B))*L;

LengthVec=[0];
L=[0];
clear B;
clear I;

k %print the number of the run
```

```
toc
end

IntEigVals=(1/nsamp)*IntEigVals; %Normalization by # of runs
NumRealEigVals= (1/nsamp)*NumRealEigVals;

% Theoretical density
[I,D,ID]=POD(1000,h,delta);
[J,W]=Weyl(1000,h);

figure
subplot(1,2,1)
scatter(BoxVec,[0,NumRealEigVals])
hold on
plot(I,D,'r')
subplot(1,2,2)
scatter(BoxVec,[IntEigVals])
hold on
plot(J,W,'r')
plot(I,ID,'g')
```

The functions *POD* and *Weyl* are given by the following programs:

```
% Weyl law for Hager's model
% the argument x denotes the number of points used
% to discretize the interval [0,1]
%
function [I,W]=Weyl(x,h)

%
D=[]; %Weyl density
W=[]; %Integrated Weyl density
%specifies points
invN=1/x;
I=[invN:invN:(1-invN)]; % discrete points, the imaginary part of "Eig"
xm=[0]; %x_+
xp=[0]; %x_-

ImDergXp=[0]; %Img'(x_+)
ImDergXm=[0]; %Img'(x_-)

k=1; %counter
%h=2*10^(-3); %Semiclassical parameter
%SQ=4/length(I);

while (k<=(x-1))
    xp(k)=asin(-I(k));
    xm(k)=pi-xp(k);
    ImDergXp(k)= -cos(xp(k));
    ImDergXm(k)= -cos(xm(k));

    D(k)=(1/(2*h*pi))*((1/ImDergXm(k))-(1/ImDergXp(k)));

    if (k==1)
        W(k)=D(k);
    else
        W(k)=W(k-1)+(1/2)*(D(k)+D(k-1));
    end
    k=k+1;
end
```

```

W=(1/W(x-1))*W;

end
```

```
% Function to plot the theoretical first density of eigenvalues
% of Hager type operators

function [I,D,ID]=POD(x,h,delta)

%Declare Variables
invN=0;
xm=[0]; %x_+
xp=[0]; %x_-
Emp_quadratic=[0]; %E_{-+}
ImDergXp=[0]; %Img'(x_+)
ImDergXm=[0]; %Img'(x_-)
FirstDensTerm=[0]; % Weyl law
SecDensTerm=[0]; %Second Density term
Density=[0]; %Density
ID=[0]; %Integrated Density
%semiclassical parameter
k=1; %counter
%h=2*10^(-3); % Semiclassical parameter
%delta=2*10^(-12); %coupling constant

% creating points
invN=1/x;
I=[invN:invN:(1-invN)]; %discrete points
SQ=4/length(I);

while (k<=(length(I)))
    xp(k)=asin(-I(k));
    xm(k)=pi-xp(k);
    ImDergXp(k)= -cos(xp(k));
    ImDergXm(k)= -cos(xm(k));
    f(k)= (-ImDergXp(k)*ImDergXm(k))^(1/2);

    Emp_quadratic(k)=(f(k)/pi)*exp((-2/h)*(-I(k)*(2*xp(k)+pi)-ImDergXp(k)
   +ImDergXm(k)));
    FirstDensTerm(k)=(1/(2*h))*((1/ImDergXm(k))-(1/ImDergXp(k)));
    SecDensTerm(k)=(Emp_quadratic(k)/(h*delta^2))*(abs(2*xp(k)+pi))^2;
    Density(k)=(1/pi)*(FirstDensTerm(k)+SecDensTerm(k))*exp(-(h*
   Emp_quadratic(k)/delta^2));

    if (k==1)
        ID(k)=SQ*Density(k);
    else
        ID(k)=ID(k-1)+(SQ/2)*(Density(k)+Density(k-1));
    end
    k=k+1;
end

M=ID(length(I));
%Normalize the density
D=(M^(-1))*Density;
ID=(M^(-1))*ID;
%D=Density;
%plot(I,D);
end
```

Figure 1.9 and the right hand side of Figure 1.3 present a comparison between the theoretical (obtained in Theorem 1.2.12) and experimental average density and average integrated density of eigenvalues of a random perturbation of the discretization of $hD+\mathrm{e}^{-ix}$ with the coupling constant δ being exponentially small in h. These figures were obtained using the following MATLAB code:

```
%Discretization of hD+exp(-ix) on the Fourier side
%This program will plot the experimental average density and the
%average integrated density
clear global;
clear;

tic %Time
% Calculate Dimension
N=1000;
D=2*N+1;

h=0.05; % Semiclassical parameter for exp small delta
delta=exp(-1/h); % Perturbation coupling constant
nsamp=400;  % Number of Runs
k=1;        % counter

E=[];
E1=[];
TempEigVals=[];
boxlength = 10^(-3);
BoxVec = [0:boxlength:1];
nBox = length(BoxVec);
LengthVec =[0];
NumRealEigVals = [0];
IntEigVals = [0];
L=[0];
HR=[]; %Perturbed Matrix

%Define unperturbed semiclassical Matrix
H=h*sparse(1:(D),1:(D),(1:(D))-N-1) + sparse(1:2*N,2:D,1,D,D);

% Experiments
for k = 1:(nsamp)
    HR=H+delta*(1/sqrt(2))*(randn(D)+1i*randn(D)); %Random Gaussian
   matrix, Exp 0, Var 1
    E=eig(HR); %find eigenvalues
    HR=[];
    E1 = [real(E) imag(E)]; % Store temporary

    %Restriction of real part and imaginary part
 for j = 1:length(E1)
        if (abs(E1(j,1))<= 45)
            if (E1(j,2)>= 0)
            TempEigVals=[TempEigVals;[E1(j,1),E1(j,2)]];
            end
        end
 end% END LOOP

E1 = TempEigVals;
TempEigVals =[];

%Sorting
[B,I] = sort(E1(:,2));
```

```
E1=[];

% Histogram Data
m=1;
for l = 1:nBox
    if ( m < length(B) )
      while ( B(m)<=BoxVec(l) )
            if (m < length(B) )
            m=m+1;
            else
                break
            end
      end
     LengthVec(l) = m;
    else
     LengthVec(l) = m;
    end
end

for j = 1:(nBox-1)
    if (LengthVec(j+1) ~= 0)
        L(j) = (4*boxlength)^(-1)*(LengthVec(j+1)-LengthVec(j));
    else
        L(j) = 0;
    end
end

%Add number of Eigvals to old, normalized by total Mass in Strip
IntEigVals = IntEigVals + (1/length(B))*LengthVec;
NumRealEigVals = NumRealEigVals + (1/length(B))*L;

LengthVec=[0];
L=[0];
clear B;
clear I;

k %print the number of the run
toc
end

IntEigVals=(1/nsamp)*IntEigVals; %Normalization by # of runs
NumRealEigVals= (1/nsamp)*NumRealEigVals;

% Theoretical density
[I,D,ID]=POD(1000,h,delta);
[J,W]=Weyl(1000,h);

figure
subplot(1,2,1)
scatter(BoxVec,[0,NumRealEigVals])
hold on
plot(I,D,'r')
subplot(1,2,2)
scatter(BoxVec,[IntEigVals])
hold on
plot(J,W,'r')
plot(I,ID,'g')
```

A.2 | Simulations for Jordan Block matrices

Figures 1.11 and 1.12 were obtained using the following MATLAB code:

```
%Jordan Block perturbed with random Gaussian Matrix
clear global;
clear;

%Initialize Variables
%N=0; % number of non zero eigenvalues of D
delta=0; % Perturbation coupling constant
D=0; %Matrix dimension

%Input Parameters
%N=input('value of N? ')
delta=input('value of delta? ')

%Calculate Dimension
%D=2*N+1;
D=500;

%Define Matrices
%HD=sparse(1:(D),1:(D),(1:(D))-N-1);
HC=sparse(1:(D-1),2:D,1,D,D);
R=(1/sqrt(2))*(randn(D) +1i*randn(D));
%Random Gaussian matrix, complex Gaussian, Expectation 0, Variance 1
H=HC+delta*R;

%Clear Unused Variables
clear HC;
clear R;

%Calculate eigenvalues
%E1=eig(full(HC));
E=eig(H);

%Plot
figure
plot(E,'o')
axis equal;
hold on             % These 3 lines to plot
plot(0,0,'r*')      % also the spectrum of the
hold off            % unperturbed Jordan block
filename='JordanBlock';
print(fig,'-dpdf',filename);
```

Figures 1.13 and 1.14 were obtained using the following MATLAB code:

```
% This program compares the experimental average density and average
    integrated
% density of eigenvalues of a randomly perturbed Jordan block matrix
%
clear global;
clear;

tic %Time
N=500;
D=2*N+1;  % Matrix Dimension
delta=2*10^(-10);   % Perturbation coupling constant
nsamp=500;  % Number of Runs
```

```
k=1;         % counter
r=1-1/D; % Cut-off radius

E=[];
TempEigVals=[];
boxlength = 1/(2*D);
BoxVec = [0:boxlength:1];
nBox = length(BoxVec);
LengthVec =[0];
NumRealEigVals = [0];
IntEigVals = [0];
L=[0];
HR=[];

%Define unperturbed Jordan block matrix
H=sparse(1:(D-1),2:D,1,D,D);

% Experiment
for k = 1:(nsamp)
    HR=H+delta*(1/sqrt(2))*(randn(D)+1i*randn(D)); %Random Gaussian
   matrix, Exp 0, Var 1
    E=eig(HR); %find eigenvalues
    HR=[];

% Restriction to smaller disk
    for j = 1:length(E)
         if (abs(E(j))< r)
             TempEigVals=[TempEigVals;[abs(E(j))]];
         end
    end% END LOOP

E = TempEigVals;
TempEigVals =[];

%Sorting

[B,I] = sort(E);
%R=real(EigVals);
%EigVals = [R(I) B];

E=[];

% HistogramData
m=1;
for l = 1:nBox
    if ( m < length(B) )
      while ( B(m)<=BoxVec(l) )
            if (m < length(B) )
            m=m+1;
            else
                break
            end
      end
     LengthVec(l) = m;
    else
      LengthVec(l) = m;
    end
end

for j = 1:(nBox-1)
```

```
    if (LengthVec(j+1) ~= 0)
        L(j) = ((2*pi*boxlength)^(-1))*(LengthVec(j+1)-LengthVec(j)); % Radial density
    else
        L(j) = 0;
    end
end

%Normalization by total Mass in Strip
IntEigVals = IntEigVals + (1/length(B))*LengthVec;
NumRealEigVals = NumRealEigVals + (1/length(B))*L;

LengthVec=[0];
L=[0];

clear B;
clear I;

k %print the number of the run
toc
end

IntEigVals=(1/nsamp )*IntEigVals; %Normalization by # of runs
NumRealEigVals= (1/nsamp )*NumRealEigVals;

% Theoretical density
%k=min(r,0.9);
[I,J,H,ID]=PHV(1000,r);

figure
subplot(1,2,1)
scatter(BoxVec,[0,NumRealEigVals])
hold on
plot(J,H,'r')
subplot(1,2,2)
scatter(BoxVec,[IntEigVals])
hold on
plot(I,ID,'r')
```

The function *PHV* in the above code is given by the following program:

```
% This functions plots the hyperbolic volume as a function of
% the radius of the unit disk
% The argument x of the functions is the number discret points in the
% interval [0,r] for the evaluation of the density

function [I,J,H,ID]=PHV(x,r)
%
H=[]; %Hyperbolic volume density
ID=[]; % Integrated density
invN=1/x; %specifies points
K=min(r,(1-invN));
J=[];
I=[0:invN:K]; % discrete points, the imaginary part of "Eig"
k=1; %counter
CutOff=160;
norm=2*I(length(I))^2/(1-I(length(I))^2);

while (k<=length(I))
    H(k)=(norm^(-1))*((2*pi)^(-1))*I(k)*(2/(1-I(k)^2))^2; %density
```

```
    ID(k) = (norm^(-1))*2*I(k)^2/(1-I(k)^2); %integrated density
    if (H(k)<=CutOff)
       J(k)=I(k);
    else
    H(k)=0; % ELse cut off
    end
    k=k+1;
end

H=H(1:length(J));

end
```

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